A_k Generalization of the $O(1)$ Loop Model on a Cylinder: Affine Hecke Algebra, q-KZ Equation and the Sum Rule

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Abstract

We study the A_k generalized model of the $O(1)$ loop model on a cylinder. The affine Hecke algebra associated with the model is characterized by a vanishing condition, the cylindric relation. We present two representations of the algebra: the first one is the spin representation, and the other is in the vector space of states of the A_k generalized model. A state of the model is a natural generalization of a link pattern. We propose a new graphical way of dealing with the Yang-Baxter equation and q-symmetrizers by the use of the rhombus tiling. The relation between two representations and the meaning of the cylindric relations are clarified through the rhombus tiling. The sum rule for this model is obtained by solving the q-KZ equation at the Razumov-Stroganov point.

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Contents

1 Introduction

The ground state of the $O(1)$ loop model (or the Temperley-Lieb (TL) stochastic process) has been extensively studied since the observation by Razumov and Stroganov [1] (see also [2, 3, 4, 5, 6, 7, 8]). Through those studies, different research areas in mathematics and physics make contact each other; for example, alternating sign matrices (ASMs) in combinatorics [9, 10] (see [11] and references therein), polynomial representations of the Temperley-Lieb/Hecke algebra in representation theory [12, 13, 14, 15], and exactly solvable models such as the six-vertex model in statistical physics (for example [16, 17]).

Razumov and Stroganov submitted seven conjectures related to the XXZ spin chain model at the anisotropic parameter $\Delta = -1/2$ with periodic conditions in [1]. These conjectures were generalized to the $O(1)$ loop model in [5]. A typical one is the sum rule (Conjecture 8 in [5]): the 1-sum of the ground state wavefunction of the $O(1)$ loop model with periodic boundary conditions and length $L = 2n$ is given by the total number of $n \times n$ ASMs. The ground state wavefunction Ψ has another remarkable property. In [3], it is conjectured that an entry Ψ_π of

Ψ is equal to the total number of fully packed loops with a link pattern π. This is the most popular form of the Razumov-Stroganov (RS) conjectures. The above mentioned sum rule is a consequence of the RS conjectures for entries.

The sum rule for the $O(1)$ loop model with periodic boundary conditions was proved by Di Francesco and Zinn-Justin by introducing inhomogeneity and utilizing the integrability [18]. The key is the q-Kniznik-Zamoldchikov (q-KZ) equation [19], which is equivalent to finding the eigenvector of the transfer matrix with eigenvalue unity. At the RS point, *i.e.* $q = -\exp(\pi i/3)$, by solving the q-KZ equation it was found that the sum of the entries of Ψ is equal to the partition function of the six-vertex model with domain wall boundary conditions (6V/DWBC). If we take the homogeneous limit where all the inhomogeneous spectral parameters z_i's tend to unity, the partition function of 6V/DWBC is proportional to the total number of ASMs [10, 20].

There are two natural generalizations of the $O(1)$ loop model. The first one is to change the geometry of the states, or equivalently to change the boundary conditions [21]. The TL algebra naturally acts on the space of link patterns. The link patterns considered in [18] are undirected ones. On the other hand, we may impose the direction on link patterns like those in [22]. The former ones are the same as link patterns with periodic boundary conditions or on an unpunctured disc, whereas the latter ones are the same as link patterns with cylindric boundary conditions or on a punctured disc, *i.e.* on a cylinder. The direction of link detects the position of the punctured point. The space of the link patterns with periodic (resp. cylindric) boundary conditions is also equivalent to the space of Dyck paths or restricted (resp. unrestricted) paths of the IRF model. The second is to extend the affine TL algebra to the affine Hecke algebra. The A_k generalized model defined in terms of the Hecke algebra of type A [23] is a natural generalization of the $O(1)$ loop model with periodic boundary conditions. Higher-rank models with open boundaries were also discussed in [24].

Remarkably, the eigenvector of the transfer matrix of the $O(1)$ loop model with periodic boundary conditions constitutes a special polynomial representation of the affine TL algebra [13]. This correspondence was also studied for the $O(1)$ model with cylindric boundary conditions in [14]. On the other hand, the special polynomial solutions of q-KZ equation for the higher-rank case of $U_q(\widehat{sl_k})$ was constructed in [15].

In this paper, we define and study the A_k generalized model on a cylinder. This model is a new hybrid generalization of the $O(1)$ loop model; defined in terms of the affine Hecke algebra of type A and with cylindric boundary conditions. The affine Hecke algebra satisfies new vanishing conditions, which we call "the cylindric relations" (see Eqn.(9) and (10) in Section 2.1). The cylindric relations fix the spin representation of the affine Hecke algebra. The intuitive meaning of the cylindric relation is to assign to a "band" around the cylinder a certain weight written in terms of the second kind of the Chebyshev polynomials. This is a natural generalization for the affine TL algebra considered in [22], where the weight of a loop around the cylinder is τ. We establish an explicit way of constructing states of the A_k generalized model. Each state is written in terms of the affine Hecke algebra and characterized by a path. For this purpose, we introduce a novel graphical way of depicting states by the use of the rhombus tiling. Although similar graphs appeared in the IRF model and the paper [23], our graphical way has the following properties. A rhombus represents the $\check{R}$-matrix constructed from the affine Hecke generator. On its face, a rhombus has a positive integer indicating the spectral parameter of the $\check{R}$-matrix. The Yang-

Baxter equation is realized as the equivalence between two different ways of tiling of a hexagon. The q-symmetrizer Y_k of the affine Hecke algebra is expressed as a $2(k+1)$-gon. A state is identified with a path via the graphical representation of the rhombus tiling. Roughly speaking, piling rhombus tiles over the $2(k+1)$-gons and reading the path on the top of the rhombus tiling, we have an unrestricted path. Indeed, an unrestricted path gives a representation of a state of our model.

We consider the eigenvector of the transfer matrix of the A_k generalized model with eigenvalue unity at the Razumov-Stroganov point, $q = -\exp(\pi i/(k+1))$. At this point, the eigenvector of the transfer matrix satisfies the q-KZ equation. Originally, the q-KZ equation has two parameters q and s. The parameter s indicates the action of the cyclic transformation. In our situation, however, the definition of the A_k generalized model has only q and assumes $s = 1$. Due to the condition $s = 1$, the eigenvector of the transfer matrix with eigenvalue unity should coincide with the solution of the q-KZ equation only at the RS point.

By resolving the solution of the q-KZ equation at the RS point, we find that the sum of the weighted entries is the product of k Schur functions. Our sum rule contains the sum rule for the $O(1)$ loop model on a cylinder when $k = 2$ [22]. Compared with the results in [15] ([14] for $k = 2$), the solution of the q-KZ equation obtained in this paper is identified with the one of level $1 + \frac{1}{k} - k$.

This paper is organized as follows. In Section 2, we briefly review the affine Hecke algebra. We introduce a class of the affine Hecke algebra which is characterized by the cylindric relation. The graphical definition and some basic properties of a rhombus with an integer are given. In Section 3, we consider the spin representation of the affine Hecke algebra. We show that the affine Hecke generator is obtained by twisting the standard Hecke generator by a diagonal matrix. Most parts of Section 3 are devoted to the proof of the cylindric relations in the spin representation. In Section 4, we move to the A_k generalized model on a cylinder. We first briefly introduce the $O(1)$ loop model on a cylinder with the perimeter of even length and reproduce the sum rule in Section 4.1.3. The derivation of the sum rule is different from [22] in the sense that we consider only the even case. We consider the space of link patterns which the affine Temperley-Lieb algebra acts on. We also explicitly write down the word representation of the highest weight state. Then, we obtain the sum rule for the $O(1)$ loop model by solving the q-KZ equation. In Section 4.2, we introduce the A_k generalization of the $O(1)$ loop model on a cylinder. We construct the states for this model through the correspondence among an unrestricted path, a rhombus tiling and a word. The relation to the spin chain model is also stated in Section 4.2.2. We solve the q-KZ equation and obtain the sum rule in Section 4.2.3. This solution is identified with the solution of the q-KZ equation of level $1 + \frac{1}{k} - k$ in Section 4.3. Section 5 is devoted to the evaluation of the recursive relation for the Schur function appeared in Section 4. Concluding remarks are in Section 6. In Appendix A.1, we show that some coefficients $C_{i,\pi,\pi'}$ (see Section 4) are equal to 1. In Appendix A.2, we give examples how the affine Hecke algebra acts on a state in the case of $(k, n) = (3, 1)$ and $k = 4$.

2 Affine Hecke Algebra

We introduce the affine Hecke algebra of type A in Section 2.1. We impose a new vanishing condition on the affine Hecke algebra, which we call the "cylindric relations". In Section 2.2 and 2.3, we present basic properties of q-symmetrizers and the Yang-Baxter equation. In Section 2.4, a rhombus with an integer is introduced for a new graphical method. The graphical ways of the Yang-Baxter equation and q-symmetrizers are given by rhombus tiling.

2.1 Affine Hecke algebra

The Iwahori-Hecke algebra $H_N(\tau)$ has generators $\{e_1, \cdots, e_{N-1}\}$ which satisfy the following defining relations:

$$e_i^2 = \tau e_i, \tag{1a}$$
$$e_i e_{i\pm1} e_i - e_i = e_{i\pm1} e_i e_{i\pm1} - e_{i\pm1}, \tag{1b}$$
$$e_i e_j = e_j e_i, \quad \text{if } |i-j| > 1, \tag{1c}$$

where we set $\tau = -(q+q^{-1})$. If we set $t_i = e_i + q$, the Iwahori-Hecke algebra can be regarded as the quotient algebra of the braid group: $t_i t_j = t_j t_i$ for $|i-j| > 1$, $t_i t_{i\pm1} t_i = t_{i\pm1} t_i t_{i\pm1}$ and the quotient relation $(t_i - q)(t_i + q^{-1}) = 0$.

When the algebra $H_N(\tau)$ satisfies the vanishing condition

$$Y_k(e_i, \cdots, e_{i+k-1}) = 0, \quad \text{for } i = 1, \cdots, N-k, \tag{2}$$

we denote this $U_q(\mathfrak{sl}(k))$ quotient Hecke algebra by $H_N^{(k)}(\tau)$. Here, the relation Y_m is the Young's q-symmetrizer, defined recursively as

$$Y_{m+1}(e_i, \cdots, e_{i+m}) = Y_m(e_i, \cdots, e_{i+m-1})(e_{i+m} - \mu_m) Y_m(e_i, \cdots, e_{i+m-1}) \tag{3}$$

with $Y_1(e_i) = e_i$ where $\mu_m = \frac{U_{m-1}(\tau)}{U_m(\tau)}$ and $U_m := U_m(\tau)$ is the Chebyshev polynomials of the second kind subject to $U_m(2\cos x) = \frac{\sin(m+1)x}{\sin x}$. The explicit expression of U_m is given by

$$U_m(\tau) = (-)^m \frac{q^{m+1} - q^{-(m+1)}}{q - q^{-1}}, \tag{4}$$
$$\mu_m = -\frac{q^m - q^{-m}}{q^{m+1} - q^{-(m+1)}}. \tag{5}$$

In particular, $H_N^{(2)}(\tau)$ is the Temperley-Lieb algebra.

The affine Hecke algebra is an extension of the Iwahori-Hecke algebra, obtained by adding the generators $y_i, 1 \le i \le N$. The generators satisfy (1) and

$$y_i y_j = y_j y_i \tag{6}$$
$$t_i y_j = y_j t_i \qquad \text{if } j \neq i, i+1 \tag{7}$$
$$t_i y_{i+1} = y_i t_i^{-1} \qquad \text{if } i \le N-1. \tag{8}$$

Let us introduce the cyclic operator σ through Yang's realization of the affine relation. Then, y_n is obtained from the recursive relation $y_n = t_{n-1}^{-1} y_{n-1} t_{n-1}^{-1}$ with $y_1 = t_1 t_2 \cdots t_{n-1}\sigma$.

We may define an additional generator $t_N = \sigma t_1 \sigma^{-1}$, or $e_N = \sigma e_1 \sigma^{-1}$. Note that the cyclic operator σ makes the defining relations (1) become cyclic and it holds the relations $\sigma t_i = t_{i-1}\sigma$ for all i. In what follows, we mainly focus on the generators $\{e_1, \dots, e_N, \sigma\}$, since the affine Hecke algebra can be constructed from these generators.

In this paper, we consider the case where N is a multiple of k, *i.e.*, $N = nk$ with an positive integer n and also consider the special case of the affine Hecke algebra $\widehat{H_N^{(k)}}(\tau)$ by imposing the additional vanishing condition as follows.

- When $N = k$, we have

$$Y_{k-1}(e_1, \cdots, e_{N-1})(e_N - \tau) Y_{k-1}(e_1, \cdots, e_{N-1}) = 0. \tag{9}$$

  Obviously, $Y_k(e_1, \cdots, e_N)$ is non-zero.

- For $N = nk$ with $n \geq 2$

$$Y_{q\text{-sym}} \cdot \prod_{i=1}^{n-1} (e_{ik} - \mu_{k-1})(e_{nk} - \tau) \cdot Y_{q\text{-sym}} = 0 \tag{10}$$

  where $Y_{q\text{-sym}} := \prod_{i=0}^{n-1} Y_{k-1}(e_{ik+1}, \cdots, e_{(i+1)k-1})$ is the product of the q-symmetrizers.

Below we call these vanishing conditions as the *cylindric relations*.

Remark1 When $n \geq 2$, the quotient relation (2) also becomes cyclic. When $n = 1$, the cylindric relation can be regarded as a modified quotient relation. The reason why the vanishing condition (2) breaks will become clear when we consider the spin representation in Section 2.

Remark2 If we set $\sigma = t_{N-1}^{-1} \cdots t_1^{-1}$, we obtain the affine Hecke algebra considered in [13, 22].

2.2 Basic properties of q-symmetrizer

For later convenience, we abbreviate $Y_k(e_i, \cdots, e_{i+k-1})$ as $Y_k^{(i)}$. Then, we have

Proposition 2.1. *The q-symmetrizer satisfies the following properties:*

- $e_j Y_k^{(i)} = Y_k^{(i)} e_j = \tau Y_k^{(i)}$ *if* $i \leq j \leq i+k-1$,
- *When* $l \leq k$ *and* $i \leq j \leq i+k-l$, *we have* $Y_k^{(i)} Y_l^{(j)} = Y_l^{(j)} Y_k^{(i)} = \alpha_l Y_k^{(i)}$ *where* $\alpha_l = \prod_{i=1}^{k} \mu_i^{-2^{k-i}}$.

Proof. We use the method of induction. When $k = 1$, we have $e_i Y_1^{(i)} = Y_1^{(i)} e_i = e_i^2 = \tau e_i$ from the definition of $Y_1^{(i)}$. We assume that the statement holds true for less than or equal to n, i.e. $e_j Y_n^{(i)} = \tau Y_n^{(i)}$ for $i \leq j \leq i+n-1$. From the definition $Y_{n+1}^{(i)} = Y_n^{(i)}(e_{i+n} - \mu_n) Y_n^{(i)}$, we have $e_j Y_{n+1}^{(i)} = e_j Y_n^{(i)}(e_{i+n} - \mu_n) Y_n^{(i)} = \tau Y_{n+1}^{(i)}$ for $i \leq j \leq i+n-1$. Since $Y_l^{(j)}$ consists of the generators

$e_i, \cdots, e_{i+l-1}$, we also have $Y_l^{(j)} Y_n^{(i)} = \alpha_l Y_n^{(i)}$ for $l \leq n$ and $i \leq j \leq i+n-l$. Then, the action of e_{i+n} on $Y_{n+1}^{(i)}$ is calculated as

$$\begin{aligned}
e_{i+n} Y_{n+1}^{(i)} &= e_{i+n} Y_{n-1}^{(i)} (e_{i+n-1} - \mu_{n-1}) Y_{n-1}^{(i)} (e_{i+n} - \mu_n) Y_n^{(i)} \\
&= Y_{n-1}^{(i)} e_{i+n} (e_{i+n-1} - \mu_{n-1})(e_{i+n} - \mu_n) Y_{n-1}^{(i)} Y_n^{(i)} \\
&= \alpha_{n-1} Y_{n-1}^{(i)} e_{i+n} (e_{i+n-1} - \mu_{n-1})(e_{i+n} - \mu_n) Y_n^{(i)} \\
&= \tau \alpha_{n-1} Y_{n-1}^{(i)} (e_{i+n-1} - \mu_{n-1})(e_{i+n} - \mu_n) Y_n^{(i)} \\
&= \tau Y_{n+1}^{(i)}
\end{aligned}$$

where we have used in the last equality the relations $e_{i+n-1} Y_n^{(i)} = \tau Y_n^{(i)}$, $e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1}$ and $\tau \mu_j - \mu_j \mu_{j-1} = 1$ for any j. We also have $Y_n^{(i)} e_j = \tau Y_n^{(i)}$ in the similar way. The proof of the first property is then completed.

We consider the case $Y_n = Y_n^{(1)}$ for the second relation. We have $Y_k \cdot Y_k = \alpha_k Y_k$ from the definition of α_k. On the other hand, by using inductive relations, we have

$$\begin{aligned}
Y_k \cdot Y_k &= Y_{k-1}(e_{k-1} - \mu_{k-1}) Y_{k-1} Y_{k-1} (e_{k-1} - \mu_{k-1}) Y_{k-1} \\
&= \alpha_{k-1} Y_{k-1} (e_k - \mu_{k-1}) Y_{k-2} (e_{k-1} - \mu_{k-2}) Y_{k-2} (e_k - \mu_{k-1}) Y_{k-1} \\
&= \alpha_{k-1} \alpha_{k-2}^2 Y_{k-1} (e_k - \mu_{k-1})(e_{k-1} - \mu_{k-2})(e_k - \mu_{k-1}) Y_{k-1} \\
&= \alpha_{k-1} \alpha_{k-2}^2 \frac{1}{\mu_k \mu_{k-1}} Y_k.
\end{aligned}$$

The proof is completed by checking α_n the recurrence relation,

$$\alpha_n = \frac{1}{\mu_n \mu_{n-1}} \alpha_{n-1} \alpha_{n-2}^2, \tag{11}$$

with the initial condition $\alpha_1 = \tau = \mu_1^{-1}$ and $\alpha_2 = \tau(\tau^2 - 1) = \mu_1^{-2} \mu_2^{-1}$, is satisfied by

$$\alpha_n = \prod_{i=1}^{n} \mu_i^{-2^{n-i}}. \tag{12}$$

□

2.3 $\check{R}$-matrix and the Yang-Baxter equation

2.3.1 $\check{R}$-matrix and the Yang-Baxter equation

Let $\check{R}_{ii+1} = e_i + q$ be the $\check{R}$-matrix where e_i is the generator of $\widehat{H_N^{(k)}}$. One can show that $\check{R}_{ii+1}$ satisfies the braid relation

$$\check{R}_{ii+1} \check{R}_{i+1i+2} \check{R}_{ii+1} = \check{R}_{i+1i+2} \check{R}_{ii+1} \check{R}_{i+1i+2}. \tag{13}$$

$\check{R}_{ii+1}$ has the inverse $\check{R}_{ii+1}^{-1} = e_i + q^{-1}$ from the *unitary* relation $\check{R}_{ii+1} \check{R}_{ii+1}^{-1} = \check{R}_{ii+1}^{-1} \check{R}_{ii+1} = 1$.

Let us introduce the permutation $\mathcal{P}_{ij}$ which exchange the indices i and j. The R-matrix is defined by $R_{ii+1} = \check{R}_{ii+1}\mathcal{P}_{ii+1}$ and obeys the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{14}$$

The trigonometric $\check{R}$-matrix can be constructed through the Baxterization of the $\check{R}$-matrix:

$$\check{R}_i(u) = \frac{1}{\zeta(u)}\left(u^{1/2}\check{R}_{ii+1} - u^{-1/2}\check{R}^{-1}_{ii+1}\right) \tag{15}$$

where $\zeta(u) = qu^{-1/2} - q^{-1}u^{1/2}$. The inverse of $\check{R}_i(u)$ is given by $\check{R}_i^{-1}(u) = \check{R}_i(u^{-1})$ and satisfies the *unitary* relation $\check{R}_i(u)\check{R}_i(u^{-1}) = 1$.

If we rewrite Eqn.(15) in terms of the affine Hecke algebra, we have

$$\check{R}_i(z,w) := \check{R}_i\left(u = \frac{z}{w}\right) = \frac{qz - q^{-1}w}{qw - q^{-1}z}\mathbf{1} + \frac{z-w}{qw - q^{-1}z}e_i. \tag{16}$$

The Yang-Baxter equation for $R_i(u) = \check{R}_i(u)\mathcal{P}$ is written as

$$\check{R}_{ii+1}\left(\frac{z}{w}\right)\check{R}_{i+1i+2}(z)\check{R}_{ii+1}(w) = \check{R}_{i+1i+2}(w)\check{R}_{ii+1}(z)\check{R}_{i+1i+2}\left(\frac{z}{w}\right). \tag{17}$$

2.3.2 q-Symmetrizers in terms of $\check{R}$

We will show that the q-symmetrizers Y_k can be expressed in terms of the $\check{R}$-matrix. For later convenience, we define

$$\check{L}_i(m) := \frac{1}{\mu_{m+1}}\check{R}\left(\frac{z}{w} = q^{-2m}\right) = e_i - \mu_m \tag{18}$$

for $m \in \mathbb{N} = \{1, 2, \ldots\}$.

The recursive relation of the q-symmetrizer Y_{m+1} (3) is rewritten as

$$Y_{m+1}(e_i, \cdots, e_{i+m}) = Y_m(e_i, \cdots, e_{i+m-1})\check{L}_{i+m}(m+1)Y_m(e_i, \cdots, e_{i+m-1}). \tag{19}$$

with $Y_1(e_i) = \check{L}_i(1)$. From Eqn.(19), a q-symmetrizer is written as a product of $\check{L}_i$'s.

The Hecke relation $e_ie_{i\pm1}e_i - e_i = e_{i\pm1}e_ie_{i\pm1} - e_{i\pm1}$ is rewritten in terms of the q-symmetrizer as $Y_2(e_i, e_{i\pm1}) = Y_2(e_{i\pm1}, e_i)$. This relation is equivalent to the Yang-Baxter equation (17) with a specialization of the spectral parameters. The Yang-Baxter equation in terms of $\check{L}$-matrix is written as

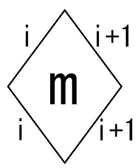
$$\check{L}_{ii+1}(u-v)\check{L}_{i+1i+2}(u)\check{L}_{ii+1}(v) = \check{L}_{i+1i+2}(v)\check{L}_{ii+1}(u)\check{L}_{i+1i+2}(u-v). \tag{20}$$

where $u, v \in \mathbb{N}$. The Hecke relation is obtained by setting $(u, v) = (2, 1)$.

2.4 Graphical representation of q-symmetrizers

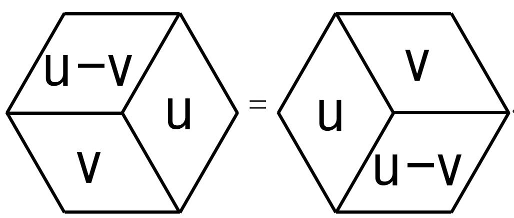
In this subsection, we introduce the graphical representation of the Yang-Baxter equation (20). Then, we also consider the graphical representation of the q-symmetrizers by using the notation used in the above subsection. It is well-known that the Yang-Baxter equation for the IRF model (see for example [25]) can be expressed as the equivalence between different rhombus tilings of a hexagon. In our novel graphical depiction, the Yang-Baxter equation is also expressed as the equivalence of two hexagons. There are, however, nice features in our method. A rhombus represents the $\check{R}$-matrix. An integer on the face of the rhombus indicates the spectral parameter of the $\check{R}$-matrix. Furthermore, the q-symmetrizers are expressed as polygons. These properties play an important role when we construct a state of the A_k generalized model in Section 4.2.1.

The Yang-Baxter equation A rhombus represents a $\check{L}$-matrix having the spectral parameter on the face of it:

$\check{L}_i(m) =$  (21)

We call the edges of a rhombus in Eqn.(21) the i-th and $(i+1)$-th edges. The i-th and $(i+1)$-th edges indicate the index i of $\check{L}_i$. We call the two corners put between i-th and $(i+1)$-th edges up and down corners, whereas the other two corners right and left corners. We may omit the name of edges without any confusion.

Accordingly, the Yang-Baxter equation (20) of the Hecke type is graphically expressed as

 (22)

Note that the $\check{L}_{ii+1}$-matrix acts on the i-th and $(i+1)$-th edges. The order of piling rhombi from the bottom corresponds to the order of $\check{L}$ from right to left (since we consider the left ideal later). A rhombus with $u-v$ in l.h.s. of Eqn.(22) is piled in the way that the bottom i-th and $(i+1)$-th edges of the rhombus are attached to the i-th and $(i+1)$-th edges of the other two rhombi.

The q-symmetrizer Y_k As mentioned above, the q-symmetrizer Y_2 is obtained by choosing the special values of the spectral parameters in the Yang-Baxter equation. From the fact that each side of the Yang-Baxter equation is graphically expressed as a rhombus tiling of a hexagon, Y_2 is also expressed graphically as the hexagon. More generally, we will see that the q-symmetrizer Y_k has a graphical representation by a $2(k+1)$-gon.

Without loss of generality, we consider $Y_k = Y_k^{(1)}$. We rewrite the q-symmetrizer Y_k as

$$
\begin{aligned}
Y_k &= Y_{k-1}\check{L}_k(k)Y_{k-1} \\
&= \alpha_{k-2}Y_{k-2}\check{L}_{k-1}(k-1)\check{L}_k(k)Y_{k-1} \\
&= C\check{L}_1(1)\check{L}_2(2)\ldots\check{L}_k(k)Y_{k-1}
\end{aligned} \tag{23}
$$

where C is a constant written in terms of α_l, $1 \leq l \leq k-1$.

Eqn.(23) implies that Y_k is obtained by piling k rhombi corresponding to a sequence of $\check{L}$ over a $2k$-gon corresponding to Y_{k-1}. Then, we have a $2(k+1)$-gon for the q-symmetrizer Y_k.

Example: The q-symmetrizers Y_3 and Y_4 are expressed as an octagon and a decagon, respectively.

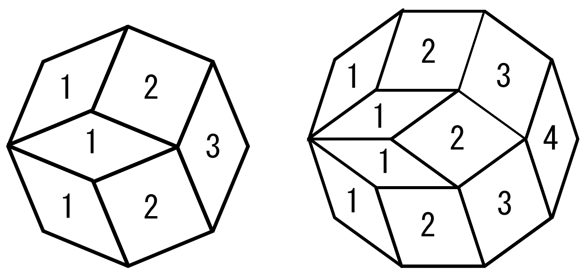


Figure 1: The graphical representation of the q-symmetrizers Y_3 and Y_4.

Equivalent expressions of Y_{k-1} We have seen that the q-symmetrizer Y_{k-1} corresponds to a rhombus tiling of a $2k$-gon. However, we have many other ways of equivalent rhombus tiling of the $2k$-gon under a sequence of *elementary moves* of rhombi. There are two equivalent ways of rhombus tiling of a hexagon as in Eqn.(22). An elementary move is an operation which changes a way of tiling from l.h.s to r.h.s and vice versa in Eqn.(22). Figure 2 shows all equivalent rhombus tilings of an octagon for the q-symmetrizer Y_3.

3 Spin Representation

In this section, we will consider the spin representation of the affine Hecke algebra $\widehat{H_N^{(k)}}$ with the cylindric relation (9) or (10). We first introduce the well-known spin representation of the Hecke algebra [26, 27], then introduce the affine generator by the twist. We show that the obtained spin representation actually satisfies the defining relations of the Hecke algebras $H_N^{(k)}(\tau)$ and the cylindric relations.

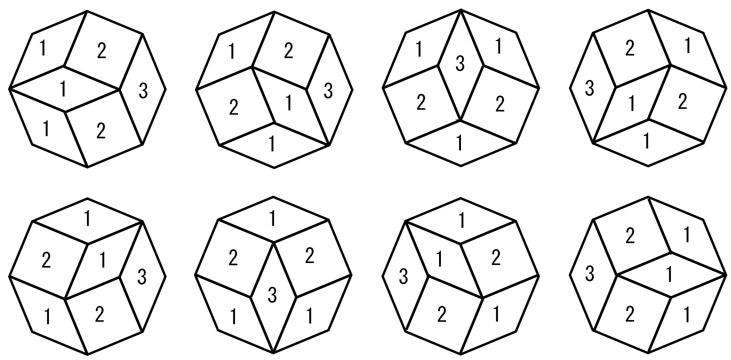


Figure 2: Equivalent expressions of the q-symmetrizer Y_3. Elementary moves of rhombus tilings are realized by the Yang-Baxter equation (22).

3.1 Hecke algebra

We first consider the spin representation of the Hecke algebra $H_N^{(k)}(\tau)$ and show the quotient relation.

Let us consider a representation $(\chi, V^{\otimes N})$ of the quotient Hecke algebra $H_N^{(k)}(\tau)$ where $\chi : H_N^{(k)}(\tau) \to \mathrm{End}(V^{\otimes N})$ and $V \cong \mathbb{C}^k$ is a vector space with the standard orthonormal basis $\{|i\rangle | 1 \leq i \leq k\}$. We denote $|i_1\rangle \otimes |i_2\rangle \otimes \cdots |i_n\rangle \in V^{\otimes n}$ by $|i_1 i_2 \cdots i_n\rangle$ for brevity. $\langle i|$ is the dual base of $|i\rangle$ with the inner product $\langle i|j\rangle = \delta_{ij}$.

We introduce $\check{e} \in \mathrm{End}(V^{\otimes 2})$ which acts on $|ij\rangle$ as

$$\check{e}|ij\rangle = (1-\delta_{ij})((-q)^{\mathrm{sign}[j-i]}|ij\rangle + |ji\rangle), \tag{24}$$

or equivalently, in terms of the standard basis of $\mathfrak{gl}_k$:

$$\check{e} = \sum_{a,b=1}^{k} E_{ab} \otimes E_{ba} - \sum_{a,b=1}^{k} q^{\mathrm{sign}[b-a]} E_{aa} \otimes E_{bb} \tag{25}$$

where E_{ab} is a $k \times k$ matrix whose elements are $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$.

A generator of the Hecke algebra has a representation of $\mathrm{End}(V^{\otimes N})$ and is written in terms of $\check{e}$:

$$\chi(e_i) = \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{i-1} \otimes \check{e} \otimes \underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{N-i-1} \tag{26}$$

where $\mathbb{I}$ is the $k \times k$ identity matrix. Below, we write as e_i instead of $\chi(e_i)$.

It is straight forward to show that e_i satisfies the defining relations of the Hecke algebra (1). We need to show that this representation actually satisfies the quotient relation.

Proposition 3.1. *([27]) In the representation $(\chi, V^{\otimes N})$, the generators e_i's satisfy the quotient relation*

$$Y_k(e_i, \cdots, e_{i+k-1}) = 0, \quad \text{for } i = 1, \cdots, N-k. \tag{27}$$

Proof. Let us denote $Y_k(e_1, \cdots, e_k)$ by $Y_k^{(1)}$ and $|v\rangle = |v_1 \cdots v_{k+1}\rangle \otimes |w\rangle \in V^{\otimes N}$ with $|w\rangle \in V^{\otimes N-k-1}$. Since $|w\rangle$ is invariant under the action of $Y_k^{(1)}$ from the matrix representation of e_i's, it is enough to show that $Y_k^{(1)}|v\rangle = 0$ for any set $\{v_1, \cdots, v_{k+1}\}$ with $1 \le v_i \le k$. We use the method of induction. Assume that the statement is true up to $k-1$. The action of e_i, $1 \le i \le k$, on $|v\rangle$ is $e_i|v\rangle = 0$ if $v_i = v_{i+1}$. From Prop. 2.1, we have

$$Y_k^{(1)}|v\rangle = \frac{1}{\alpha_{k-1}} Y_k^{(1)} Y_{k-1}^{(1)}|v\rangle = \frac{1}{\alpha_{k-1}} Y_k^{(1)} Y_{k-1}^{(2)}|v\rangle. \tag{28}$$

Since $Y_k^{(1)}|v\rangle$ is non-vanishing if $\{v_1, v_2, \cdots, v_k\}$ and $\{v_2, \cdots, v_{k+1}\}$ are all distinct from the assumption, v_1 is to be equal to v_{k+1}. From (24), we have

$$\begin{aligned} Y_k^{(1)}|v\rangle &= \frac{1}{\alpha_1} Y_k^{(1)} e_1 |v\rangle \\ &= \frac{1}{\alpha_1} Y_k^{(1)}((-q)^{\mathrm{sign}[v_2-v_1]}|v\rangle + |v_2 v_1 v_3 \cdots v_{k+1}\rangle \otimes |w\rangle \\ &= \frac{(-q)^{\mathrm{sign}[v_2-v_1]}}{\alpha_1} Y_k^{(1)}|v\rangle \end{aligned}$$

where we have used $Y_{k-1}^{(2)}|v_2 v_1 v_3 \cdots v_{k+1}\rangle \otimes |w\rangle = 0$. Since $\frac{(-q)^{\mathrm{sign}[v_2-v_1]}}{\alpha_1} \neq 1$ for general q ($q \neq 0$), we have $Y_k^{(1)}|v\rangle = 0$. □

The q-symmetrizer satisfies the following properties in the spin representation. Below, we restrict the action of $Y_{k-1}^{(i)}$ to $W = V^{\otimes k}$ where $V^{\otimes N} = V^{\otimes i-1} \otimes W \otimes V^{\otimes N-i-k}$, since $Y_{k-1}^{(i)}$ acts as identity except on W.

Proposition 3.2. *For a given k, the q-symmetrizer $Y_{k-1}^{(i)}$ has only one eigenvector (up to normalization) with a non-zero eigenvalue in $V^{\otimes k}$ and its eigenvalue is α_{k-1} given by (12).*

Proof. We will show that we have a simultaneous eigenvector of e_i's and that is also the only eigenvector of the q-symmetrizer. From the spin representation, we only need to prove that $Y_{k-1} = Y_{k-1}^{(1)}$ has only one eigenvector with a non-zero eigenvalue. Let $v = |v_1 \cdots v_k\rangle$ be a vector in $V^{\otimes k}$. From Proposition 3.1, a non-vanishing v satisfies $v_i \neq v_j$ for any i, j. We denote by $\widetilde{V^{\otimes k}}$ the subspace of $V^{\otimes k}$ where $1, \cdots, k$ appear exactly once in $\{v_1, \cdots, v_k\}$. Note that non-vanishing eigenvectors of Y_{k-1} are in $\widetilde{V^{\otimes k}}$. From Proposition 2.1, eigenvectors of Y_{k-1} are also simultaneous eigenvectors of all the Hecke generators e_i's and vice versa. The eigenvectors of e_i with non-zero eigenvalue have the form

$$|\cdots v_i v_{i+1} \cdots\rangle + (-q)^{\mathrm{sign}[v_i - v_{i+1}]}|\cdots v_{i+1} v_i \cdots\rangle. \tag{29}$$

Starting from $|v\rangle = |v_1 \cdots v_k\rangle = |12 \ldots k\rangle \in \widetilde{V^{\otimes k}}$, we may fix the simultaneous eigenvector of all e_i's (up to the overall constant) as follows:

$$|v_0\rangle = \sum_{s \in \mathfrak{S}_k} (-q)^{l(s)} |s(v)\rangle, \tag{30}$$

where $v_0 \in \widetilde{V^{\otimes k}}$, $\mathfrak{S}_k$ is the symmetric group and $|s(v)\rangle = |v_{s(1)}v_{s(2)}\cdots v_{s(k)}\rangle$. The function $l(s)$ satisfies $l(ss^{'}) = l(s) + l(s^{'})$ and

$$l(s_{ii+1}) = \begin{cases} 1, & v_i > v_{i+1} \\ -1, & v_{i+1} > v_i \end{cases}, \tag{31}$$

where $s_{ii+1} \in \mathfrak{S}_k$ is the transposition between v_i and v_{i+1}. Another choice of $|v\rangle$ gives just the difference of overall normalization constant. Since v_0 is constructed as the simultaneous eigenvector of e_i's, v_0 is also the eigenvector of Y_k. The uniqueness of the eigenvector is guaranteed by construction.

The eigenvalue of $|v_0\rangle$ with respect to e_i is τ. Then, the action of Y_{k-1} on $|v_0\rangle$ is $Y_{k-1}|v_0\rangle = \tilde{y}_{k-1}|v_0\rangle$ where $\tilde{y}_{k-1} = Y_{k-1}(\tau, \cdots, \tau)$ is a c-number. $\tilde{y}_{k-1}$ satisfies the same recurrence relation and the initial condition as (11). This means $\tilde{y}_{k-1} = \alpha_{k-1}$. □

For later convenience, we write down the inner product of $|v_0\rangle$ (the eigenvector of Y_{k-1} with non-zero eigenvalue) in terms of the Chebyshev polynomials of the second kind.

Corollary 3.3. *Let $|v_0\rangle$ be the eigenvector of the q-symmetrizer Y_k with the non-zero eigenvalue. The inner product $I_k = \langle v_0|v_0\rangle$ in the representation $(\chi, V^{\otimes N})$ is calculated as $q^{-k(k-1)/2}\prod_{i=1}^{k} U_i$.*

Proof. From Proposition 3.2, the eigenvector $|v_0\rangle$ can be rewritten as

$$|v_0\rangle = \sum_{s\in\mathfrak{S}_k}(-q)^{l(s)}|s(v)\rangle \tag{32}$$

$$= \sum_{1\leq v_1\leq k}\sum_{\tilde{s}\in\mathfrak{S}_{k-1}}(-q)^{-(v_1-1)+l(\tilde{s})}|v_1\tilde{s}(v\setminus v_1)\rangle \tag{33}$$

where $v = |12\dots k\rangle$ and $|v_1\tilde{s}(v\setminus v_1)\rangle = |v_1 v_{\tilde{s}(2)}\cdots v_{\tilde{s}(k)}\rangle$. The inner product I_k is then calculated as

$$\begin{aligned} I_k &= \langle v_0|v_0\rangle = \sum_{s\in\mathfrak{S}_k}(-q)^{2l(s)} \\ &= \sum_{1\leq v_1\leq k} q^{-2(v_1-1)}\sum_{\tilde{s}\in\mathfrak{S}_{k-1}} q^{2l(\tilde{s})} = (-q)^{-(k-1)}U_{k-1}I_{k-1} \\ &= (-q)^{-k(k-1)/2}\prod_{1\leq i\leq k-1} U_i. \end{aligned} \tag{34}$$

□

3.2 Affine Hecke algebra

Let us introduce a linear operator $\tilde{e} \in \mathrm{End}(V\otimes V)$ in the basis of $\mathfrak{gl}_k$ by

$$\tilde{e} = \sum_{a,b=1}^{k} q^{2(a-b)}E_{ab}\otimes E_{ba} - \sum_{a,b=1}^{k} q^{\mathrm{sign}[b-a]}E_{aa}\otimes E_{bb}. \tag{35}$$

This $\tilde{e}$ is obtained by the twist, *i.e.*, $\tilde{e} = \Omega^{-1}\breve{e}\Omega$ where the twist Ω is given by $\Omega = \mathbb{I} \otimes \tilde{\Omega}$, $\tilde{\Omega} = \mathrm{diag}(q^{-(k-1)}, q^{-(k-3)}, \cdots, q^{k-1})$.

It is also straightforward to show that the spin representation (35) of $\tilde{e}$ satisfies the following two properties aff1-2.

(aff1) $\tilde{e}^2 = \tau\tilde{e}$

(aff2) $\tilde{e}_{12}e_{23}\tilde{e}_{12} - \tilde{e}_{12} = e_{12}\tilde{e}_{23}e_{12} - e_{12}$

where $\tilde{e}$ is a $k \times k$ matrix and $\tilde{e}_{12} = \tilde{e} \otimes \mathbb{I}$, $\tilde{e}_{23} = \mathbb{I} \otimes \tilde{e}$ and $\tilde{e}_{13} = \mathcal{P}_{23}\tilde{e}_{12}\mathcal{P}_{23}$ ($\mathcal{P}$ is a permutation matrix).

Now we are ready to construct an additional generator e_N which allows us to have the affine Hecke algebra. Let us introduce the shift operator ρ acting on the basis in $V^{\otimes N}$. Let $v = |v_1 \cdots v_N\rangle$ be a base in $V^{\otimes N}$. Then, $\rho : V^{\otimes N} \to V^{\otimes N}$ is defined by $\rho : v \mapsto |v_2 \cdots v_N v_1\rangle$. We define e_N acting on $V^{\otimes N}$ as

$$e_N = \rho^{-1}(\underbrace{\mathbb{I} \otimes \cdots \otimes \mathbb{I}}_{N-2} \otimes \tilde{e})\rho. \tag{36}$$

By construction, the defining relations (1) of the Hecke algebra become cyclic and there exists the cyclic operator σ such that $\sigma e_i = e_{i-1}\sigma$ for any $i \in \mathbb{Z}/N\mathbb{Z}$. The cyclic operator σ in the spin representation is explicitly given by

$$\sigma = (\Omega\rho)^{-1}. \tag{37}$$

Here $\Omega = \mathbb{I}^{\otimes N-2} \otimes \tilde{\Omega}$. Note that σ^N is the identity. In this way, we construct the affine Hecke generators $\{e_1, \cdots, e_N, \sigma\}$ in the spin representation.

Remark: There may be other linear operators in $\mathrm{End}(V \otimes V)$ which satisfy the properties aff1 and aff2. If we set $\sigma = \rho^{-1}$ instead of (37), we also have another affine Hecke algebra. However, this algebra does not satisfy the cylindric relation (may satisfy another kind of vanishing condition).

The affine algebra $\widehat{H_N^{(k)}}$ can be regarded as a natural generalization of the affine TL algebra on a cylinder. As we will see in the next paragraph, the affine Hecke algebra $\widehat{H_N^{(k)}}$ satisfies the cylindric relation introduced in the Section 2. To relate this affine Hecke algebra to the loop models, we need to have a further condition for $\tilde{e}$, which comes from the weight of a "loop" surrounding a cylinder. Although the graphical way to describe a "loop" model corresponding to the A_k-vertex model is not known as far as the authors know, it is natural to assign that the weight of a "loop" is related to the Chebyshev polynomials of the second kind. This is realized by the cylindric relation (see Section 4.2).

Let us consider the case $k = 2$. The spin representation of $\tilde{e}$ is given by

$$\tilde{e} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & q^{-2} & 0 \\ 0 & q^2 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

Note that $\tilde{e} = \Omega^{-1}\breve{e}\Omega$ where $\Omega = \mathbb{I} \otimes \tilde{\Omega}$ and $\tilde{\Omega} = \mathrm{diag}(q^{-1}, q)$, that is, $\tilde{e}$ is the twist of $\breve{e}$. This $\tilde{e}$ is equivalent to the boundary considered in [6, 22].

Cylindric relation and extra vanishing conditions We need to show that the above spin representation satisfies the cyclicity of the vanishing conditions (2) and the cylindric relation (9) or (10).

The vanishing conditions (2) of the quotient Hecke algebra become cyclic by adding the extra generator e_N when $n \geq 2$. We omit the proof but similar to the Proposition 3.1 because of the properties aff1 and aff2.

We will show that the spin representation satisfies the cylindric relations (9) or (10) in the following two propositions.

Proposition 3.4. *In the representation* $(\chi, V^{\otimes N})$ *of the affine Hecke algebra, the cylindric relation for* $N = k$ *is given by*

$$Y_{k-1}(e_1, \cdots, e_{k-1})(e_k - \tau)Y_{k-1}(e_1, \cdots, e_{k-1}) = 0. \tag{39}$$

Proof. By using Proposition 3.2 and 3.3, the q-symmetrizer $Y^{(1)}_{k-1}$ is written as

$$Y_{k-1} = \frac{\alpha_{k-1}}{A_{k-1}}|v_0\rangle\langle v_0| \tag{40}$$

where α_{k-1} is given by (12) and $A_{k-1} = (-q)^{-k(k-1)/2} \prod_{1\leq i\leq k-1} U_i$.

Equation (39) is rewritten as $\langle v_0|e_k|v_0\rangle = \tau\langle v_0|e_k|v_0\rangle$. By the definition $|v_0\rangle = \sum_{s\in\mathfrak{S}_k}(-q)^{l(s)}|s(v)\rangle$, we have

$$\langle v_0|e_N|v_0\rangle = \sum_{s,s'\in\mathfrak{S}_k}(-q)^{l(s)+l(s')}\langle s'(v)|e_k|s(v)\rangle. \tag{41}$$

Due to the representation (35) of e_k, the expectation value $\langle s'(v)|e_k|s(v)\rangle$ is shown to be non-zero for either $s' = s$ or $s' = s_{1k}s$ where s_{1k} is the transposition operator. Therefore, we have

$$l(s') = \begin{cases} l(s), & s' = s \\ l(s) - 2(v_{s(k)} - v_{s(1)}) + \mathrm{sign}[v_{s(k)} - v_{s(1)}], & s' = s_{1k}s \end{cases} \tag{42}$$

and

$$\frac{\langle s'(v)|e_k|s(v)\rangle}{\langle s'(v)|s(v)\rangle} = \begin{cases} (-q)^{\mathrm{sign}[v_{s(1)}-v_{s(k)}]}, & s' = s \\ (-q)^{2(v_{s(k)}-v_{s(1)})}, & s' = s_{1k}s \end{cases} \tag{43}$$

Substituting Eqn.(43) and Eqn.(42) into Eqn.(41), we finally obtain

$$\begin{aligned} \langle v_0|e_k|v_0\rangle &= \sum_{s,s'\in\mathfrak{S}_k}(-q)^{l(s)+l(s')}\langle s'(v)|e_k|s(v)\rangle \\ &= \sum_{s\in\mathfrak{S}_k}(-q)^{2l(s)}(-q-q^{-1})\langle s(v)|s(v)\rangle \\ &= \tau\langle v_0|v_0\rangle. \end{aligned} \tag{44}$$

□

Proposition 3.5. *In the case of $N = nk$, $n \geq 2$, the affine Hecke generators $\{e_1, \cdots, e_N\}$ in the representation $(\chi, V^{\otimes N})$ satisfy the following relation:*

$$Y_{q\text{-sym}} \cdot \prod_{i=1}^{n-1} (e_{ik} - \mu_{k-1}) \cdot (e_{nk} - \tau) \cdot Y_{q\text{-sym}} = 0, \tag{45}$$

where $Y_{q\text{-sym}} = \prod_{i=0}^{n-1} Y_{k-1}^{(ik+1)}$.

Proof. We first rewrite the relation (45) into a simpler form.

Let $|v_0^{(i+1)}\rangle = \sum_{s\in\mathfrak{S}_k}(-q)^{l(s)}|v_{s(1)}^{(i)} \cdots v_{s(k)}^{(i)}\rangle \in V^{\otimes k}$ be the eigenvector of $Y_{k-1}^{(ik+1)}$ for $0 \leq i \leq n-1$ (see Proposition 3.2). The unique eigenvector $|v_0\rangle \in V^{\otimes N}$ of $Y_{q\text{-sym}}$ is given by the tensor product of $|v_0^{(i)}\rangle$, *i*.e. $|v_0\rangle = \bigotimes_{i=0}^{n-1}|v_0^{(i+1)}\rangle$. By taking the expectation value w.r.t. $|v_0\rangle$, Eqn.(45) is rewritten as

$$\Big\langle \prod_{i=1}^{n-1} (e_{ik} - \mu_{k-1}) \cdot (e_{nk} - \tau) \Big\rangle = 0, \tag{46}$$

where we denote $\langle v_0|\mathcal{O}|v_0\rangle$ for some operator $\mathcal{O}$ by $\langle \mathcal{O}\rangle$.

The vanishing condition $Y_k^{(ik+1)} = 0$ for any $i \in \mathbb{Z}/k\mathbb{Z}$ is expressed as $\langle e_{(i+1)k} - \mu_k\rangle = 0$. In general, we have

$$\Big\langle \prod_{j=1}^{m} (e_{i_m k} - \mu_k) \Big\rangle = 0 \tag{47}$$

for $1 \leq i_m \leq k$ and $1 \leq m \leq n-1$ where we have used the relation $e_{lk} Y_{k-1}^{(ik+1)} = Y_{k-1}^{(ik+1)} e_{lk}$ for $l \neq ik, (i+1)k$. Equation (46) can be rewritten as

$$\begin{aligned} \text{l.h.s. in (46)} &= \Big\langle \prod_{i=1}^{n} (e_{ik} - \mu_k) + \Delta_k^{n-1}(\mu_k - \tau) \Big\rangle && (48)\\ &= \Big\langle \prod_{i=1}^{n} e_{ik} - \mu_k^n + \Delta_k^{n-1}(\mu_k - \tau) \Big\rangle && (49) \end{aligned}$$

where we have used Eqn.(47), $\langle \prod_{i=1}^{m} e_{ik}\rangle = \mu_k^m$, $1 \leq m \leq k-1$, and $\Delta_k = \mu_k - \mu_{k-1}$. By the relations $\tau - \mu_k = \mu_{k+1}^{-1}$ and $U_{k-1}^2 - U_k U_{k-2} = 1$, eventually the wanted relation equivalent to Eqn.(45) is equivalent to

$$\Big\langle \prod_{i=1}^{n} e_{ik} \Big\rangle = \frac{1}{U_k^n U_{k-1}^n}(U_{k-1}^{2n} + U_{k-1}U_{k+1})\langle v_0|v_0\rangle. \tag{50}$$

On the other hand, we can evaluate $\langle \prod_{i=1}^{n} e_{ik}\rangle$ by using the spin representation. Let us consider the action of $\prod_{i=1}^{n} e_{ik}$ on the vector $|v\rangle = |v_1^{(1)} \ldots v_k^{(1)} v_1^{(2)} \ldots v_k^{(n)}\rangle \in \left(\widetilde{V^{\otimes k}}\right)^{\otimes n}$. The operator e_{ik} acts locally on $v_k^{(i)}$ and $v_1^{(i+1)}$. To have a non-vanishing expected value, $\langle v'|\prod_{i=1}^{k} e_{ik}|v\rangle \neq 0$, an admissible $\langle v'|$ satisfies $v'^{(i)}_m = v_m^{(i)}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, $2 \leq m \leq k-1$ and either of the following conditions:

- $v'^{(i)}_1 = v^{(i)}_1$ and $v'^{(i)}_k = v^{(i)}_k$ for all $i \in \mathbb{Z}/n\mathbb{Z}$,
- $v'^{(i)}_k = v^{(i+1)}_1$ and $v'^{(i+1)}_1 = v^{(i)}_k$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Now, we are ready to evaluate

$$\left\langle \prod_{i=1}^{n} e_{ik} \right\rangle = \sum_{S} (-q)^{\sum_i l(s_i)+l(t_i)} \Big\langle \bigotimes_{i=1}^{n} t_i(v^{(i)}) \Big| \prod_{i=1}^{n} e_{ik} \Big| \bigotimes_{i=1}^{n} s_i(v^{(i)}) \Big\rangle. \tag{51}$$

where the sum S is take all over $s_1, \cdots, s_n \in \mathfrak{S}_k$ and $t_1, \cdots, t_n \in \mathfrak{S}_k$. From the above considerations, we split the calculation into two cases as follows.

Case 1: We consider the case where $s_i(1) = s_j(1), s_i(k) = s_j(k)$ for all $i, j \in \mathbb{Z}/n\mathbb{Z}$ and t_i's satisfy

- $t_i(1) = s_{i-1}(k)$ and $t_i(k) = s_{i+1}(1)$ for all $i \in \mathbb{Z}/n\mathbb{Z}$,
- $t_i(j) = s_i(j)$ for $2 \leq j \leq k-1$ and all $i \in \mathbb{Z}/n\mathbb{Z}$.

We abbreviate as $t = t_i$, $s = s_i$ without confusion. From (31), we have

$$l(t) = l(s) - 2(v_{s(k)} - v_{s(1)}) + \mathrm{sign}[v_{s(k)} - v_{s(1)}]. \tag{52}$$

If we rewrite $|s(v)\rangle$ in terms of $\tilde{s} \in \mathfrak{S}_{k-2}$ such that $|s(v)\rangle \propto |v_{s(1)}\tilde{s}(v)v_{s(k)}\rangle$, we have

$$l(s) = l(\tilde{s}) + (v_{s(k)} - v_{s(1)}) - (k-2) - \frac{1}{2}\mathrm{sign}[v_{s(k)} - v_{s(1)}] - \frac{1}{2}. \tag{53}$$

The action of $\prod_{i=1}^{n-1} e_{ik}$ gives a factor 1 and that of e_{nk} gives q^a where

$$a = -2(v^{(1)}_{s(1)} - v^{(n)}_{s(k)}). \tag{54}$$

Substituting Eqn.(52), (53) and (54) into (51), we have

$$\begin{aligned} J_1 &= \sum_{1 \leq v^{(n)}_{s(k)} \neq v^{(1)}_{s(1)} \leq k} \ \sum_{\substack{\tilde{s}_i \in \mathfrak{S}_{n-2} \\ 1 \leq i \leq n}} (-q)^{-n(2k-3) + \sum_i 2l(\tilde{s}_i) - 2(v^{(1)}_{s(1)} - v^{(n)}_{s(k)})} \\ &= (-q)^{-n(2k-3)} I^n_{k-2}(U_{k+1}U_{k-1} - (k-1)) \\ &= \frac{1}{U^n_k U^n_{k-1}} (U_{k+1}U_{k-1} - (k-1)) \langle v_0 | v_0 \rangle \end{aligned} \tag{55}$$

where we have used the recurrence relation for I_k obtained in Proposition 3.3.

Case 2: We consider the case where $s_i(j) = t_i(j)$ for all $1 \leq j \leq k$ and $i \in \mathbb{Z}_n$. Obviously, we have

$$l(t_i) = l(s_i) \tag{56}$$

for all i. The action of e_{ik}'s gives a factor $(-q)^b$ where

$$b = \sum_{i=1}^{n} \mathrm{sign}[v_{s(1)}^{(i+1)} - v_{s(k)}^{(i)}] \tag{57}$$

Substituting (53), (56) and (57) into (51), we have

$$\begin{aligned} J_2 &= \sum_{s_i \in \mathfrak{S}_n} (-q)^{\sum_i 2l(s_i) + \mathrm{sign}[v_{s(1)}^{(i+1)} - v_{s(k)}^{(i)}]} \\ &= \sum_{S'} \sum_{\tilde{s}_i \in \mathfrak{S}_{k-2}} (-q)^{\sum_i 2l(\tilde{s}_i) + 2(v_{s(k)}^{(i)} - v_{s(1)}^{(i)}) - 2(k-2) - \mathrm{sign}[v_{s(k)}^{(i)} - v_{s(1)}^{(i)}] - 1 + \mathrm{sign}[v_{s(1)}^{(i+1)} - v_{s(k)}^{(i)}]} \\ &= (-q)^{-n(2k-3)} I_{k-2} \sum_{S'} (-q)^{\sum_i 2(v_{s(k)}^{(i)} - v_{s(1)}^{(i)}) - \mathrm{sign}[v_{s(k)}^{(i)} - v_{s(1)}^{(i)}] - \mathrm{sign}[v_{s(k)}^{(i)} - v_{s(1)}^{(i+1)}]} \end{aligned} \tag{58}$$

and the sum is taken over all the sets:

$$S' = \left\{ v_{s(1)}^{(i)}, v_{s(k)}^{(i)} \;\middle|\; \begin{array}{c} 1 \leq v_{s(1)}^{(i)}, v_{s(k)}^{(i)} \leq k, \quad i = 1, 2, \ldots, n \\ v_{s(1)}^{(i)} \neq v_{s(k)}^{(i)}, v_{s(1)}^{(i)} \neq v_{s(k)}^{(i-1)} \end{array} \right\}. \tag{59}$$

From Lemma 3.6 (see below), J_2 in Eqn.(58) is rewritten as

$$J_2 = \frac{1}{U_k^n U_{k-1}^n} (U_{k-1}^{2n} + (k-1)). \tag{60}$$

Together with Eqn.(55) and Eqn.(51), we finally obtain

$$\begin{aligned} \left\langle \prod_{i=1}^{n} e_{ik} \right\rangle &= J_1 + J_2 \\ &= \frac{1}{U_k^n U_{k-1}^n} (U_{k-1}^{2n} + U_{k-1} U_{k+1}) I_k \end{aligned} \tag{61}$$

and this completes the proof of the relation (45). □

We need the following lemma.

Lemma 3.6. *Let*

$$I = \sum_{S'} q^{\sum_{l=1}^{n} 2(i_{2l} - i_{2l-1}) - \mathrm{sign}[i_{2l} - i_{2l-1}] - \mathrm{sign}[i_{2l} - i_{2l+1}]} \tag{62}$$

where $i_{2n+1} = i_1$ *and*

$$S' = \left\{ i_l \;\middle|\; \begin{array}{c} 1 \leq i_l \leq k, \\ i_l \neq i_{l\pm1}, \end{array} \text{ for } 1 \leq \forall l \leq 2n \right\}. \tag{63}$$

Then, I is calculated in terms of $U'_{k-1} = U_{k-1}(-\tau)$ *as*

$$I = (U'_{k-1})^{2n} + (k-1). \tag{64}$$

Proof. Let us introduce a set of integer variables, $U = \{(u_1, \ldots, u_{2n}) | 1 \leq u_j \leq k-1, \text{for } 1 \leq j \leq 2n\}$ and the subset of S', $S'_{extra} = \{(i_1 \ldots, i_{2n}) | i_{2l-1} = k' + 1, i_{2l} = k', 1 \leq k' \leq k-1, \text{for } 1 \leq l \leq n\}$.

We introduce the shift operator acting on a sequence of length $2n$, $S = (s_1, \cdots, s_{2n})$, by $\xi : S \to S, s_i \mapsto s_{i+1}$ for $i \in \mathbb{Z}_{2n}$. We consider two subsets $S_0 \subset S' \backslash S'_{extra}$ and $U_0 \subset U$:

$$
\begin{aligned}
S_0 &= \left\{ (i_1, \cdots, i_{2n}) \in S' \backslash S'_{extra} \,\middle|\, \xi^m(i_j) \geq (i_j), \;\; 1 \leq \forall m \leq 2n-1 \right\}, && (65)\\
U_0 &= \left\{ (u_1, \cdots, u_{2n}) \in U \,\middle|\, \xi^m(u_j) \geq (u_j), \;\; 1 \leq \forall m \leq 2n-1 \right\}. && (66)
\end{aligned}
$$

The symbol $\geq$ means lexicographic order, *i*.e. $\mu \geq \nu$ stands for $\mu_j = \nu_j$ for all $1 \leq j \leq 2n$, or $\mu_j = \nu_j$ for $1 \leq j \leq i$ and $\mu_{i+1} > \nu_{i+1}$ for some i. When we have a bijection $\eta : S_0 \to U_0$, We extend a bijection from $S' \backslash S'_{extra}$ to U. For a given $i \in S' \backslash S'_{extra}$, we have a non-negative integer $r_{min} = \min\{r : \xi^r i \in S_0\}$. Then a bijection is extended by $\xi^{-r_{min}} \circ \eta \circ \xi^{r_{min}}$.

We construct a bijection $\eta : S_0 \to U_0$ by first constructing an injection $\eta : S_0 \to U_0$ and showing there exists the injective inverse η^{-1}.

The map $\eta : S_0 \to U_0$ defines u_j recursively starting from u_1:

$$
\begin{aligned}
u_1 &= i_1, \qquad u_2 = u_1 + d'_1, \\
u_j &= u_{j-1} + \begin{cases} d_j, & d_j d_{j-1} > 0 \\ d'_j, & d_j d_{j-1} < 0 \end{cases}, && (67)
\end{aligned}
$$

where $d_j := i_{j+1} - i_j$ and $d'_j := i_{j+1} - i_j - \mathrm{sign}[i_{j+1} - i_j]$ for $1 \leq j \leq 2n-1$. From the construction, η is injective. We have $\mathrm{Im}(\eta) \subseteq U_0$ since the branching rule (67) assures $u_j \geq u_1$ and $\max\{u\} \leq \max\{i\} - 1 \leq k-1$. Then, the inverse $\eta^{-1} : U_0 \to S_0$ is explicitly given by

$$
\begin{aligned}
i_1 &= u_1, && (68)\\
i_j &= i_{j-1} + t_{j-1} && (69)
\end{aligned}
$$

and

$$
t_j = \begin{cases} -\mathrm{sign}[t_{j-1}], & \bar{d}_{j-1} = \bar{d}_j = 0 \\ -\mathrm{sign}[\bar{d}_{j-1}], & \bar{d}_{j-1} \neq 0, \bar{d}_j = 0 \\ \bar{d}_j + \mathrm{sign}[\bar{d}_j], & t_{j-1}\bar{d}_j > 0 \\ \bar{d}_j, & t_{j-1}\bar{d}_j < 0 \end{cases} \qquad (70)
$$

where $\bar{d}_j = u_{j+1} - u_j$ with the initial condition $t_1 = \bar{d}_1 + 1$. The map η^{-1} is also injective. It is easy to verify that $\mathrm{Im}(\eta^{-1}) \subseteq S_0$ since $i_{2n} \geq i_1 + 1$, $i_j \neq i_{j+1}$ for $1 \leq \forall j \leq 2n$ and $\max\{i_j\} = \max\{u\} + 1 \leq k$. From these, $\eta : S_0 \to U_0$ is a bijection.

Note that when $i \in S_0$ and $u = \eta(i) \in U_0$, we have

$$
\begin{aligned}
\sum_{l=1}^{n} 2(u_{2l} - u_{2l-1}) &= \sum_{l=1}^{n} (u_{2l} - u_{2l-1}) + (u_{2l} - u_{2l+1}) \\
&= \sum_{i=1}^{n} 2(i_{2l} - i_{2l-1}) - \mathrm{sign}[i_{2l} - i_{2l-1}] - \mathrm{sign}[i_{2l} - i_{2l+1}],
\end{aligned}
$$

since the branching rules (67) give the correct term for $u_{2l} - u_1 = i_{2l} - i_1 - 1$.

From these observations, we arrive at

$$\begin{aligned} I &= \sum_{U} q^{\sum_{l=1}^{2n}(u_{2l}-u_{2l-1})} + \sum_{S'_{extra}} 1 \\ &= (U'_{k-1})^{2n} + (k-1). \end{aligned} \tag{71}$$

□

Example: The following list gives some examples of the bijection.

i	u	Exponent in Eqn.(71)
$(1,2,3,7,4,2)$	$(1,1,2,6,4,2)$	4
$(7,2,3,4,2,1)$	$(6,2,2,3,2,1)$	-8
$(2,1,5,3,1,4,2,5)$	$(2,1,4,3,1,3,2,4)$	4

Remark When $N = k$, the cylindric relation can be regarded as a modification of the vanishing condition (2), although the relation (2) is no longer satisfied. However, when $n \geq 2$, the vanishing relations (2) become cyclic and the cylindric relation (45) is a highly non-trivial relation.

3.3 $\check{R}$-matrix

We give the spin representation of $\check{R}$ introduced in Section 2. The R-matrix of the $\mathfrak{gl}_k$ spin chain model is given by

$$R = q\sum_{a=1}^{k} E_{aa} \otimes E_{aa} + \sum_{1\leq a\neq b\leq k} E_{aa} \otimes E_{bb} + (q - q^{-1}) \sum_{1\leq b<a\leq k} E_{ab} \otimes E_{ba}. \tag{72}$$

We also introduce the permutation operator $\mathcal{P}$ and the q-permutation operator as follows:

$$\mathcal{P} \equiv \mathcal{P}_{12} = \mathcal{P}_{21} = \sum_{a,b=1}^{k} E_{ab} \otimes E_{ba}, \tag{73}$$

$$\begin{aligned} \mathcal{P}^q_{12} &= \sum_{a=1}^{k} E_{aa} \otimes E_{aa} + q\sum_{a<b}^{k} E_{ab} \otimes E_{ba} + q^{-1}\sum_{a>b}^{k} E_{ab} \otimes E_{ba} \\ &= \sum_{a,b=1}^{k} q^{\mathrm{sign}[b-a]} E_{ab} \otimes E_{ba}. \end{aligned} \tag{74}$$

$$(\mathbb{I}\otimes\mathbb{I})_q = \sum_{a,b=1}^{k} q^{2(a-b)} E_{aa} \otimes E_{bb}. \tag{75}$$

The Baxterized R-matrix is given by

$$R_{12}(u) = \frac{a(u)}{\zeta(u)}\mathcal{P}_{12} + \frac{b(u)}{\zeta(u)}(\mathbb{I}\otimes\mathbb{I} - \mathcal{P}^q_{12}) \tag{76}$$

where $a(u) = qu^{1/2} - q^{-1}u^{-1/2}$ and $b(u) = u^{1/2} - u^{-1/2}$. Note that $\check{R} = R\mathcal{P}$. We also have

$$R_{N1}(u) = a(u)\mathcal{P}_{N1} + b(z)((\mathbb{I}\otimes\mathbb{I})_q - \mathcal{P}^q_{N1}) \tag{77}$$

corresponding to the affine generator e_N. The R-matrix (and $\check{R}$-matrix) satisfies $R_{i+1i+2} = \sigma^{-1}R_{ii+1}\sigma$ for $1 \leq i \leq N-1$ and $R_{NN+1} = R_{N1}$.

4 A_k Generalized Model on a Cylinder

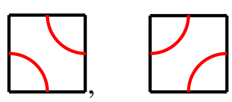
We first briefly review the $O(1)$ loop model on a cylinder, which is the $k = 2$ case of the A_k generalized model on a cylinder in Section 4.1. We introduce a new model which we call A_k generalized model on a cylinder in Section 4.2. The way of constructing states is given by the use of the rhombus tiling. The relation of our model to the spin chain model is stated in Section 4.2.2. We obtain the sum rule of the A_k generalized model by solving the q-KZ equation at the Razumov-Stroganov point in Section 4.2.3. The solution is identified with a special solution of the q-KZ equation in Section 4.3.

4.1 $O(1)$ loop model on a cylinder

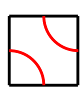
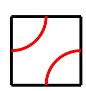
4.1.1 The $O(1)$ loop models

In this subsection, the results of the $O(1)$ loop models are presented. Here is the summary of the result if we take $k = 2$ in the Section 4.2. See also [18, 22] for some results and details.

Definition of the $O(1)$ loop models The inhomogeneous $O(1)$ loop model is defined on a semi-infinite cylinder of square lattice with even perimeter $2n$, where squares on the same height are labelled in order cyclically from 1 to $2n$. Spectral parameter z_i for $1 \leq i \leq 2n$ is attached to each vertical strip. We attach two kinds of unit plaquettes

 (78)

on the square. The weight of the plaquettes in the i-th vertical strip is given by the R-matrix as

$$R(z_i, t) = \frac{qz_i - q^{-1}t}{qt - q^{-1}z_i}$$  $$+ \frac{z_i - t}{qt - q^{-1}z_i}$$  (79)

where t is a horizontal spectral parameter.

States and the boundary condition Since red (or grey) lines on a plaquette are non-intersecting, a site is connected to another site by a non-intersecting red (or grey) line. From this, all the $2n$ sites are connected to each other forming a link. The space of states for the $O(1)$ loop model is the set of link patterns. We denote a state by π, or $|\pi\rangle$.

We introduce the direction of links and the boundary of the cylinder. Let us consider a conformal map from the semi-infinite cylinder to a disk with perimeter $2n$. The infinite point is mapped to the origin of the disk. The two boundary conditions are classified by whether we regard the origin of the disk as an punctured point or not as follows.

- Periodic boundary condition (or unpunctured case) : The infinite point is regarded as an unpunctured point. In this case, we focus only on connectivities between the sites.
- Cylindric boundary condition (or punctured case): The infinite point is regarded as a punctured point. Introducing the punctured point corresponds to introducing a seam between the first and the $2n$-th sites on the cylinder. The direction of a link between sites i and j is measured by $(-1)^w$ where w counts how many times the link crosses the seam.

We assign to a loop (even a loop surrounding the punctured point) the weight $\tau = -(q + q^{-1})$ when q is a generic value. Note that when q is a cubic root of unity, *i.e.*, $q = -\exp(\pi i/3)$, the weight of a loop is $\tau = 1$.

Transfer matrix and q-KZ equation The row-to-row transfer matrix of the $O(1)$ loop model (in both periodic and cylindric cases) is given by

$$T(t|z_1, \cdots, z_{2n}) = \mathrm{Tr}_0 \left(R_1(z_1, t)R_2(z_2, t) \dots R_{2n}(z_{2n}, t)\right) \tag{80}$$

where the trace is taken on the auxiliary quantum space. The transfer matrix naturally acts on a states. We want to compute the weight distribution $\Psi(z_1, \cdots, z_{2n}) = \sum_\pi \Psi_\pi |\pi\rangle$ such that

$$T(t|z_1, \cdots, z_{2n})\Psi(z_1, \cdots, z_{2n}) = \Psi(z_1, \cdots, z_{2n}). \tag{81}$$

Instead of the eigenvector problem (81), it is enough to consider more generally the q-KZ equation with two parameters q and s (see [13, 18])

$$\begin{aligned} \check{R}(z_i, z_{i+1})\Psi(z_1, \dots, z_{2n}) &= \tau_{i,i+1}\Psi(z_1, \dots, z_{2n}), \qquad 1 \le i \le 2n-1 \\ \check{R}(z_{2n}, sz_1)\Psi(z_1, \dots, z_{2n}) &= \Psi(s^{-1}z_{2n}, \dots, sz_1) \end{aligned} \tag{82}$$

where $\tau_{i,i+1}$ is an operator acting on a polynomial $f(z_i, z_{i+1})$ as $\tau_{i,i+1} f(z_i, z_{i+1}) = f(z_{i+1}, z_i)$. When $s = 1$, the eigenvector of the transfer matrix with eigenvalue unity is regarded as the solution of the q-KZ equation. This is realized at the Razumov-Stroganov point, *i.e*. $q = -\exp(\pi i/3)$ in the link pattern basis [18, 22]. The solution of the q-KZ equation with generic q and $s = q^6$ (resp. $s = q^3$) was obtained on the link patterns with periodic (rep. cylindric) boundary conditions in [13] (resp. [14]).

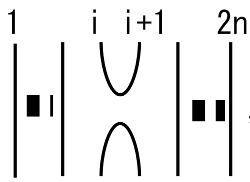
Below, we construct the space of link patterns in the cylindric case, on which the affine Temperley-Lieb algebra acts.

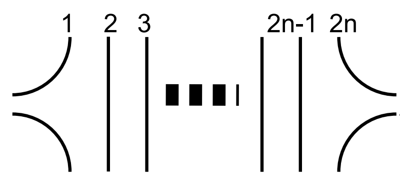
4.1.2 Word representation (cylindric case)

In this subsection, the parameter q takes a generic value.

Word representation and cylindric relation It is well-known that the word representation of link patterns (periodic case) is constructed in the left ideal of the Temperley-Lieb algebra. The lowest state ω is given by the product of q-symmetrizer Y_1, $\omega := \prod_{i=1}^{n} e_{2i-1}$. All the other states are obtained by taking actions of a sequence of e_i's, *i.e.* words.

We can construct all the states for the cylindric case in the similar way from ω. However, the additional operator e_{2n} appears in the word representation. It is natural that the graphical representation of the generators $e_i, 1 \le i \le 2n-1$, and e_{2n} of the affine Temperley-Lieb algebra is

$e_i =$ 

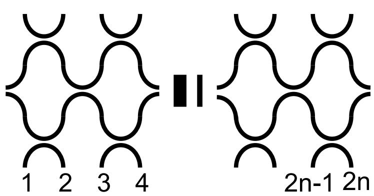
, $e_{2n} =$ 

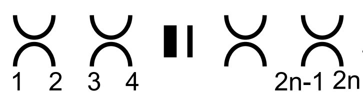
. (83)

The cylindric relation (45) can be written in terms of the affine Temperley-Lieb generators as

$$\prod_{j=1}^{n} e_{2j-1} \prod_{i=1}^{n} e_{2i} \prod_{j=1}^{n} e_{2j-1} = \tau^2 \prod_{j=1}^{n} e_{2j-1}. \tag{84}$$

This relation can be depicted using the graphical representation as



$= \tau^2$ 

. (85)

Note that the weight of a loop surrounding the cylinder is τ and the factor τ^2 in (84) comes from the two loops in the r.h.s. of Eqn.(85).

The highest state We write as $\overleftarrow{\prod_{1\le j\le r}} s_j := s_r s_{r-1} \dots s_1$ where the order of products is clear, and $(s_r \dots s_1)^{\vee(m)} := s_r \dots s_{1+(r+m)/2} s_{(r-m)/2} \dots s_1$ for $r > m \ge 0$ and $r \equiv m \pmod 2$. We define $(s_r \dots s_1)^{\vee(m)} = 1$ for $m \ge r$ and $(s_r \dots s_1)^{\vee(m)} = s_r \dots s_1$ for $m < 0$. For example, $(\overleftarrow{\prod_{1\le j\le 5}} s_j)^{\vee 3} = s_5 s_1$ and $(\overleftarrow{\prod_{1\le j\le 6}} s_j)^{\vee 2} = s_6 s_5 s_2 s_1$.

The highest state $)) \dots ((($ is given in terms of the generators as

$$\overleftarrow{\prod_{0\le j\le \lfloor n/2 \rfloor}} \tilde{E}_j \tag{86}$$

where

$$\tilde{E}_j = \begin{cases} (\overleftarrow{\prod_{1\le l\le n-1}} e_{2l})^{\vee(2j)} \cdot e_{2n} \cdot (\prod_{l=1}^{n} e_{2l-1})^{\vee(2j-1)}, & \text{for } n \text{ odd} \\ (\overleftarrow{\prod_{1\le l\le n-1}} e_{2l})^{\vee(2j+1)} \cdot e_{2n} \cdot (\prod_{l=1}^{n} e_{2l-1})^{\vee(2j)}, & \text{for } n \text{ even} \end{cases} \tag{87}$$

for $j \geq 0$. Other states are obtained as words by acting a sequence of the generators on the highest state.

Example: $n = 2$ We have six bases. The correspondence of representation by parentheses and by words is given as follows: ()() $\leftrightarrow$ e_1e_3, (()) $\leftrightarrow$ $e_2e_1e_3$,)()($\leftrightarrow$ $e_2e_4e_1e_3$, ())($\leftrightarrow$ $e_1e_2e_4e_1e_3$,)(() $\leftrightarrow$ $e_3e_2e_4e_1e_3$ and))(($\leftrightarrow$ $e_4e_1e_3$. Then, the Temperley-Lieb generators are given by

$$e_1 = \begin{pmatrix} \tau & 1 & 0 & 0 & \tau^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \sigma = \begin{pmatrix} 0 & 0 & \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 & 0 \\ \tau^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau^{-1} \\ 0 & \tau^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau & 0 \end{pmatrix}, \tag{88}$$

and $e_3 = \sigma e_2 \sigma^{-1}$ and $e_4 = \sigma e_3 \sigma^{-1}$. Note that they satisfy the defining Hecke relations and the cylindric relation $e_1e_3e_2e_4e_1e_3 = \tau^2 e_1e_3$.

Relation to the spin chain An affine Temperley-Lieb generator $e_i \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ in the spin-1/2 representation. This allows us to rewrite a link pattern in terms of the spin basis [6]. For a given directed link between the site i and j ($i < j$), a spin vector is written as

$$|\uparrow\downarrow\rangle_{ij} + (-q)^{-1}|\downarrow\uparrow\rangle_{ij}, \qquad \text{for } i \to j, \tag{89}$$

$$|\uparrow\downarrow\rangle_{ij} + (-q)^{-3}|\downarrow\uparrow\rangle_{ij}, \qquad \text{for } j \to i. \tag{90}$$

In the periodic case, every link is expressed as Eqn.(89) since we do not see the direction. In the cylindric case, however, we take a vector of the type (89) for a link uncrossing the seam of the cylinder and a vector of the type (90) for a link crossing the seam. In both cases, take the tensor product of associated vectors for a link pattern.

4.1.3 q-KZ equation and the sum rule

The q-KZ equation connects the polynomial representation of the affine Temperley-Lieb algebra and the word representation of the algebra [14, 22]. When $\tau = 1$, the q-KZ equation can be explicitly solved. Then, it is found that the sum rule for Ψ is the product of two Schur functions. When we take the homogeneous limit all $z_i \to 1$, the sum is proportional to the total number of the $2n \times 2n$ half-turn symmetric alternating sign matrices (HTSASMs) (see also [2, 4, 5, 20]).

4.2 A_k generalized model

We define the A_k generalized model on a cylinder. This is the A_k generalization of the $O(1)$ loop model on a cylinder in Section 4.1. This generalization is done by replacing the affine Temperley-Lieb algebra and state space labelled by link patterns with the affine Hecke algebra and state space labelled by unrestricted paths, respectively. In Section 4.2.1, we explicitly construct a state by the use of the graphical depiction of the q-symmetrizer introduced in Section 2.4. The relation of our model to the spin chain model is discussed in Section 4.2.2.

We set up the q-KZ equation (82) where $\check{R}_i(z, w)$ is now the standard trigonometric $\check{R}$-matrix defined in (16). We examine the q-KZ equation with two parameters $s = 1$ and $q = -\exp(\pi i/k + 1)$. The sum rule for the solution $\Psi(\mathbf{z})$ is investigated in Section 4.2.3.

4.2.1 States for A_k generalized model

Before constructing the states for the A_k generalized model, we introduce some definitions and notations. Hereafter, we set $N = nk$. The parameter q is generic in this subsection.

Definition 4.1. *We define a set of unrestricted paths and restricted paths.*

1. *An unrestricted path $\pi := \pi_1\pi_2 \cdots \pi_{nk}$ of length nk is a set of nk integers satisfying*

$$\begin{aligned} &1 \le \pi_i \le k \ \ \textit{for}\ 1 \le i \le nk \\ &\sharp\{i|\pi_i = j, 1 \le i \le nk\} = n \ \ \textit{for}\ \ 1 \le j \le k. \end{aligned} \tag{91}$$

2. *If an unrestricted path π satisfies $\sharp\{i|\pi_i = j, 1 \le i \le l\} \ge \sharp\{i|\pi_i = j + 1, 1 \le i \le l\}$ for all $1 \le j \le k$ and $1 \le l \le nk$, the path π is said to be a restricted path.*

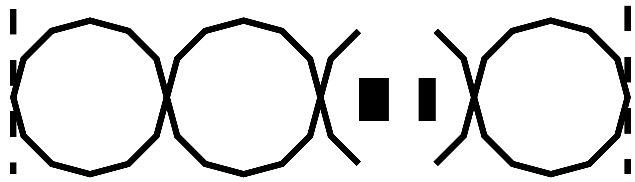
The number of unrestricted paths is $(nk)!/(n!)^k$. The number of restricted paths is the same as the number of the standard Young tableaux with shape $k \times n$, *i.e.,* $(nk)! \prod_{0 \le j \le k-1} j!/(n + j)!$.

A path is graphically depicted by a line graph on a cylinder. A line graph π' of length N consists of $N + 1$ vertex and N edges. The i-th and $(i + 1)$-th vertices are connected by the i-th edge for all $1 \le i \le N$. The first and $(N + 1)$-th vertices are identified in the case of a line graph on a cylinder. When the angular coefficient of the i-th edge is $\pi(k - 2m + 1)/2k$ with $1 \le m \le k$, the i-th edge is said to be of type m. A path π is identified with a line graph of length nk on a cylinder where the i-th edge is of type π_i.

In Section 2.4, we have seen that a q-symmetrizer Y_k corresponds to a $2(k + 1)$-gon. Recall that $Y_q\text{-sym}$ is the product of the q-symmetrizers.

Definition 4.2. *The graphical representation of $Y_q\text{-sym} = \prod_{i=0}^{n-1} Y_k(e_{ik+1}, \ldots, e_{(i+1)k})$ is the graph where we put n $2(k+1)$-gons side by side. The terminal vertices are identified as the graph is on a cylinder.*

Example The graphical expression of Y_q-sym is shown for $k = 6$. The cylinder is cut along the dotted line.

Y_q-sym =  . (92)

We have nk edges on the top of the graphical representation of Y_q-sym. From Definition 4.1 and Definition 4.2, we have a corresponding path as follows.

Proposition 4.3. *The top edges of Y_q-sym are identified with the path π^{Ω} where $\pi^{\Omega}_{pk+q} = q$ for $0 \leq p \leq n-1$ and $1 \leq q \leq k$.*

Consider a path π that satisfies $\pi_i > \pi_{i+1}$ for a given i. In this situation, we can pile a rhombus with a positive integer m, corresponding to $\check{L}_i(m)$, locally over the line graph consisting of the i-th and the $(i+1)$-th edges. Then, we obtain a new path π' satisfying

$$\begin{aligned} &\pi^{'}_{j} = \pi_j \quad \text{for } j \neq i, i+1 \\ &\pi^{'}_{i} = \pi_{i+1}, \quad \pi^{'}_{i+1} = \pi_i. \end{aligned} \tag{93}$$

where $\pi_{kn+1} = \pi_1$. We introduce an order of paths such that if (93) is satisfied the path $\pi^{'}$ is lower than π. The order of paths is the same as the one of tiling rhombi.

Definition 4.4. *We pile rhombi from bottom to top over the graphical representation of Y_q-sym, or equivalently, over the path π^{Ω}. We pile rhombi one by one following the rule in Eqn.(93).*

From the above definition, we have a natural map from a rhombus tiling over π^{Ω} to an unrestricted path. The top edges of the rhombus tiling are identified with a line graph on a cylinder and a path.

Definition 4.5. *We introduce a word and a reduced word.*

1. *Let l be a positive integer. $M = \{m_j | 1 \leq m_j \leq k-1, 1 \leq j \leq N\}$ and $I = \{i_j | 1 \leq i_j \leq N, 1 \leq j \leq l\}$ are sets of positive integers. A word w of length l is defined as*

   $$w = \check{L}_{i_1}(m_1) \dots \check{L}_{i_l}(m_l) Y_q\text{-sym} \tag{94}$$

   for some $\{l, M, I\}$ and Y_q-sym $= \prod_{i=0}^{n-1} Y_k^{(ik+1)}$. By definition, Y_q-sym is itself a word of length zero. The length of a linear combination of words is identified by the maximum length of words in it.

2. *A word w is said to be equivalent to another word w' if we obtain w from w' only by using the defining relations of the affine Hecke algebra, (1), (2) and the cylindric relations (9) or (10).*

3. *A word w (of length l) is said to be a reduced word if there exists no equivalent word of length l' with $l' < l$.*

Hereafter, a word means a reduced word.

The word representation is the representation of the affine Hecke algebra on the left ideal $\widehat{H_N^{(k)}} Y_{q\text{-sym}}$.

Remark We restrict ourselves to $1 \leq m_j \leq k-1$ in Definition 4.5. Since $\check{L}_i(m) = \check{L}_i(m-1) + (\mu_{m-1} - \mu_m)$, a word can be rewritten in terms of other words. The vanishing condition (2) and the graphical representation of a q-symmetrizer in Section 2.4 imply that $1 \leq m_j \leq k-1$ is enough to have non-vanishing words.

From these definitions, we have the following map from a rhombus tiling to a word. For a given rhombus tiling over π^{Ω}, we have a natural map from a rhombus tiling with integers to a word w as Eqn.(94).

The above definitions and proposition are summarized as follows. We have a word and an unrestricted path π for a rhombus tiling with integers. There are, however, many rhombus tilings with integers whose top edges are characterized by the path π, whereas we have only one word for a given rhombus tiling with integers.

We want to get a state $|\pi\rangle$ labelled by an unrestricted path π satisfying the following properties.

(P1) If π satisfies $\pi_i > \pi_{i+1}$, the state is invariant under under the action of e_i, *i.e.* $e_i|\pi\rangle = \tau|\pi\rangle$.

(P2) If π satisfies $\pi_i < \pi_{i+1}$, the action of e_i is given by $e_i|\pi\rangle = \sum_{\pi'} C_{i,\pi,\pi'}|\pi'\rangle$. If the coefficient $C_{i,\pi,\pi'} \neq 0$, π' is obtained by adding a unit rhombus as (93) or a path below π.

(P3) If π satisfies $\pi_i = \pi_{i+1}$, the action of e_i is given by $e_i|\pi\rangle = \sum_{\pi'} C_{i,\pi,\pi'}|\pi'\rangle$. If the coefficient $C_{i,\pi,\pi'} \neq 0$, π' is a path below π.

(P4) In the properties (P2) and (P3), let us consider the case where $e_j|\pi\rangle = \tau|\pi\rangle$ for $j \neq i \pm 1$. Then, for a path π' with non-zero $C_{i,\pi,\pi'}$ it satisfies $e_j|\pi'\rangle = \tau|\pi'\rangle$.

We will have the one-to-one correspondence;

$$\begin{array}{c}\text{a state } |\pi\rangle \\ \text{labelled by a path } \pi\end{array} \Longleftrightarrow \begin{array}{c}\text{a rhombus tiling} \\ \text{with integers}\end{array} \Longleftrightarrow \text{a word.} \tag{95}$$

A specific choice of a rhombus tiling for a given path π allows us to describe the state $|\pi\rangle$ in terms of a word. The difficulty is to assign positive integers to rhombi for a given rhombus tiling.

First of all,

Definition 4.6. *The word representation of the state $|\pi^{\Omega}\rangle$ is $Y_{q\text{-sym}}$. The graphical representaion of $|\pi^{\Omega}\rangle$ is the graph where we put n $(2k+1)$-gon with integers side by side.*

Before constructing enery state $|\pi\rangle$, we prepare some terminologies and notations. Let us start to assign integers to every corners of rhombi as follows (see also the explanation below Eqn.(21)).

Definition 4.7. *Consider a rhombus with an integer m. We assign $+m$ (resp. $-m$) to up and down (resp. right and left) corners of the rhombus.*

Definition 4.8. *Suppose that a vertex is shared by some rhombi. We assign to the vertex the sum of all the integers on corners around the vertex.*

Definition 4.9 (zero-sum rule)**.** *Suppose that a vertex is completely surrounded by rhombi. A vertex is said to satisfy the zero-sum rule if the sum of all signed integers on corners surrounding the vertex is equal to zero.*

Note that all the vertices inside $2(k+1)$-gon of a q-symmetrizer Y_k satisfy the zero-sum rule and that the vertices on the top path have integer one.

Definition 4.10. *Fix an integer $1 \leq l \leq k$. Consider l integers $1 \leq i_1 < i_2 < \cdots < i_l \leq k$. Let b^{top} be a partial path of length l satisfying $b^{top}_j = i_j$ for all $1 \leq j \leq l$. Fix a path of length l, b^{bot}, which satisfies each i_j appears once in $\{b^{bot}_i\}$, $b^{bot}_1 \neq i_1$ and $b^{bot}_l \neq i_l$. Connect i-th and $(i+l)$-th vertices by the two line graphs of b^{top} and b^{bot}. The $2l$-gon surrounded by the two line graphs is called a rhombus block $B_{i,i+l}$ surrounded by b^{top} and b^{bot}.*

Proposition 4.11. *A positive integer on every rhombus in a given rhombus block $B_{i,i+l}$ is uniquely determined by the zero-sum rule and the integers on the vertices, from the $(i+1)$-th to the $(i+l-1)$-th vertices in the top partial path b^{top}.*

Proof. Adding some rhombi to $B_{i,i+l}$ to form a $2l$-gon looking like a q-symmetrizer, it is sufficient to show that positive integers on rhombi in the $2l$-gon are uniquely determined by the condition for $B_{i,i+l}$. If we change the rhombus tiling of the q-symmetrizer by elementary moves, we have the form of the standard rhombus tiling of it as shown in Fig. 1 but integers are different. Since two edges b^{top}_i and b^{top}_{i+1} form a rhombus at the $(i+1)$-th vertex, the integer on the rhombus is the same as the integer on the $(i+1)$-th vertex in b^{top}. The integer on the next rhombus is determined by the integers on the first rhombus and on the $(i+2)$-th vertex in b^{top}. Integers on $l-1$ rhombi whose edge is the part of the top partial path b^{top} are determined one-by-one in this way. Other remained rhombi form a smaller polygon , *i.e.*, $2(l-1)$-gon. All the integers for this $2(l-1)$-gon are fixed by the zero-sum rule. Next, by elementary moves we get back to the equivalent expressions of $2l$-gon with integers, one of which contains the rhombus block $B_{i,i+l}$. By taking away certain pieces of rhombi from $2l$-gon, we obtain the block $B_{i,i+l}$ with integers. □

Recall that we may have many ways of rhombus tilings with integers corresponding to a path π. All the unrestricted paths are obtained up to a certain finite height starting from the lowest path π^{Ω}. For a path we take a rhombus tilings with the smallest number of rhombi. However, we have many equivalent rhombus tilings because of elementary moves of rhombi.

In particular, we want a rhombus tiling representing a state $|\pi\rangle$ satisfying the properties from (P1) to (P4). This is possible by using the freedom by elementary moves of rhombi.

Now, we explain the construction of a state $|\pi\rangle$ for a given path π. The procedure is divided into three steps. First, we fix a rhombus tiling for a given path. We divide the rhombus tiling into some pieces of rhombus blocks. Secondly, we assign integers on all the rhombi for the rhombus tiling. Finally, we identify the state $|\pi\rangle$ with one of rhombus tiling with integers.

Step1: Take one of its rhombus tilings which gives a path π. Fix an order of tiling rhombi from bottom. We have a set of lower paths than π associated with this tiling. Here, we divide a given rhombus tiling into pieces of rhombus blocks.

Step1-1 Consider a convex partial path $\pi_{i,i+m} := \pi_i \ldots \pi_{i+m}$ in π satisfying $\pi_{i-1} > \pi_i < \ldots < \pi_{i+m} > \pi_{i+m+1}$ for some $m \geq 1$. Take a lower convex partial path π' as long as possible such that π' contains the partial path $\pi_{i,i+m}$. We call π' the longest convex sequence (lc-sequence) associated with $\pi_{i,i+m}$. Write down all the longest convex sequences for the path π.

Step1-2 If two lc-sequences cross at a vertex below the path π, we modify them as follows. We keep the longer lc-sequence as it is. We split the shorter lc-sequence into two parts at the crossing point, and take away the part beneath the longer one. See Fig. 2. When two crossing sequences have the same length, one of these are to be shortened in the similar way.

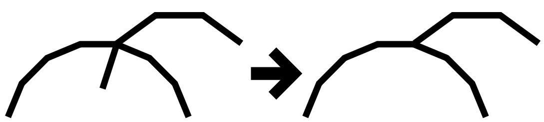


Figure 3: A part of a lc-sequence beneath a longer lc-sequence is taken away.

Step1-3 Let us denote by $\pi'_{i,i+l} = \pi'_i\pi'_{i+1}\cdots\pi'_{i+l-1}$ the lc-sequence from the i-th to the $(i+l)$-th vertex, which contains two edges of the last piled rhombus. Let $\pi'^{low}_{i,i+l}$ be a partial lower path from from the i-th vertex to $(i+l)$-th as low as possible. Then, we have a rhombus block $B_{i,i+l}$ surrounded by $\pi'_{i,i+l}$ and $\pi'^{low}_{i,i+l}$. If there is a part of another lc-sequence π_{lc} inside the block $B_{i,i+l}$, the lowest path $\pi^{low}_{i,i+l}$ is to be modified such that $\pi^{low}_{i,i+l}$ is over π_{lc}.

Step1-4 Successively, take away rhombus blocks obtained in Step1-3. Finally, we have the path π^{Ω} and many pieces of rhombus block.

(Step1 ends)

We make an order of piling rhombus blocks as follows. If two blocks are far enough, we may exchange the order of two blocks. However, if $\sharp\{k|i \leq k \leq i+l,\ j \leq k \leq j+l'\} \geq 1$ for given two blocks $B_{i,i+l}$ and $B_{j,j+l'}$, the order of two blocks is determined by the order of piling rhombi. Below, we fix an order of piling rhombus blocks. The order of removing rhombus blocks is the reverse of the piling one.

Step2: We are ready to assign positive integers to all rhombi for a given rhombus tiling.

Step2-1 Let us consider the first rhombus block in the removing order. We assign the positive integer 1 to all convex vertices on the top partial path of the block. From Proposition 4.11, we determine integers on all rhombi in this block.

Step2-2 We move to the second rhombus block. If an integer on a convex vertex in the top partial path of the second block is determined from the integers on the first block by the zero-sum rule, we assign that integer on the vertex. Otherwise, we assign 1 on them. Again, we determine integers on all rhombi in the second block from Proposition 4.11.

Step2-3 We determine integers on all subsequent rhombus blocks in the similar way. We continue this process until we assign integers on the all rhombi over π^{Ω}.

(Step2 ends)

From the construction, all the vertices inside the rhombus blocks satisfy the zero-sum rule. All the integers on convex vertices in the top path of $Y_q\text{-sym}$ are one. However, the zero-sum rule may not hold on vertices in the top path of $Y_q\text{-sym}$. This is because there is a concave vertex on the top path.

Step3: We fix integers for a given path π and its rhombus tiling with integers. Then, we choose one of rhombus tilings with integers as a state $|\pi\rangle$.

We introduce a sequence of integers $\mu = (\mu_1, \dots, \mu_k)$ for a rhombus tiling μ where μ_j is the total number of positive integer j written in rhombi. Let μ and ν be rhombus tilings corresponding to the same path, then we may have the natural order. $\mu \geq \nu$ means $\mu_k > \nu_k$, or $\mu_{k-r} = \nu_{k-r}$ for all $0 \leq r \leq i-1$ and $\mu_{k-i} > \nu_{k-i}$ for some $1 \leq i \leq k$, and $\mu = \nu$ holds when $\mu_j = \nu_j$ for all $1 \leq j \leq k$.

Definition 4.12. *We choose a rhombus tiling with μ for a path π such that $\mu \geq \nu$ for any other rhombus tiling with ν. The state $|\pi\rangle$ is identified with the word of rhombus tiling with μ.*

(Step3 ends)

Some remarks are in order.

Remark1 The cylindric relation is expressed by means of a rhombus tiling with integers. Rewrite the cylindric relations as

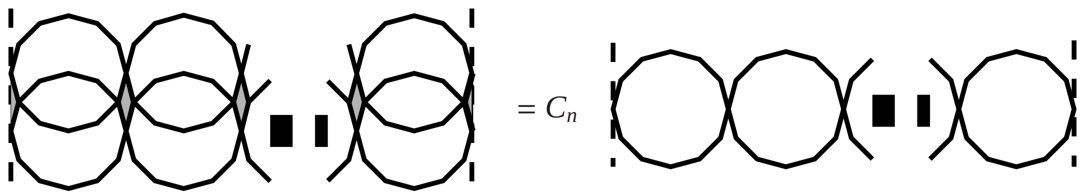
$$Y_{k-1}(e_1, \cdots, e_{k-1})\check{L}_k(k-1)Y_{k-1}(e_1, \cdots, e_{k-1}) = \mu_k^{-1}\alpha_{k-1}Y_{k-1}(e_1, \cdots, e_{k-1}) \tag{96}$$

for the case where $N = k$. Here, α_k is given by Eqn.(12). And

$$Y_q\text{-sym} \cdot \left(\prod_{i=1}^{n} \check{L}_{ik}(k-1)\right) \cdot Y_q\text{-sym} = \Delta_k^{n-1}\mu_k^{-1}\alpha_{k-1}^{n} Y_q\text{-sym} \tag{97}$$

for the case where $N = nk$ with $n \geq 2$. Here, $\Delta_k = \mu_k - \mu_{k-1}$. The graphical representation of these, which is the generalization of Eqn.(85), indicates that it is possible to truncate piles of rhombi by some height.

Example: $k = 6, n \geq 2$. We have the following graphical representation.



where $C_n = \Delta_6^{n-1}\mu_6^{-1}\alpha_5^n$, we put 5 on the grey rhombi, and the dodecagons are the q-symmetrizers Y_5. Polygons are supposed to be covered by rhombi with integers. We choose one of the equivalent expressions of the q-symmetrizer such that two dodecagons share the same rhombus of tiling of intersectional octagon with integers. The rhombus for the operator $\check{L}_N$ are divided into two parts since the cylinder is cut along the dotted line.

Remark2: We briefly explain states constructed in the above way satisfy the properties from (P1) to (P4). When an integer on a vertex in the path π is one (the vertex is convex by definition), we can bring the rhombus with one, $\check{L}_i(1) = e_i$, to the convex vertex from somewhere inside the rhombus tiling by the elementary moves of rhombi. From the first relation in Eqn.(1), we have the property (P1).

Notice that $e_i = \check{L}_i(1)$ and $\check{L}_i(m) = \check{L}_i(m-1) + (\mu_{m-1} - \mu_m)$. The action of e_i on $|\pi\rangle$ leads to getting the state π' by (93) and other states for lower paths. We have the property (P2).

Consider a path π and suppose that (P3) holds true for all lower states than π. When π satisfies $\pi_i = \pi_{i+1}$, take an action of e_i on the word for the state $|\pi\rangle$. By using the relation $e_ie_j = e_je_i$ for $|i-j| \geq 2$, we change the position of e_i from left to right as many times as possible. Then we have a word $\mathcal{L}e_i\mathcal{L}'Y_q\text{-sym}$ where $\mathcal{L}, \mathcal{L}'$ are sequences of $\check{L}$. $\mathcal{L}$ does not include e_j, $j = i, i \pm 1$ and $\mathcal{L}' = \check{L}_{i\pm1}\cdots$. The word $e_i\mathcal{L}'Y_q\text{-sym}$ is written in terms of other words whose paths are lower than that of $\mathcal{L}'Y_q\text{-sym}$. When $\mathcal{L}$ is the identity, we apply the similar argument for e_l with a certain l instead of e_j by successively using the Hecke relation $e_ie_{i\pm1}e_i - e_i = e_{i\pm1}e_ie_{i\pm1} - e_{i\pm1}$. From the assumption, $e_i\mathcal{L}'Y_q\text{-sym}$ is a linear combination of the lower states. After acting $\mathcal{L}$ on the obtained states, we get $e_i\mathcal{L}\mathcal{L}'Y_q\text{-sym}$ as a linear combination of the lower states $|\pi\rangle$.

Finally, the property (P4). Let $|\pi\rangle$ satisfy $e_j|\pi\rangle = \tau|\pi\rangle$. In the rhombus tiling, we may pile the rhombus for $\check{L}_j(1)$ at the last. In all the cases $\pi_i \gtreqless \pi_{i+1}$ for $i \neq j, j \pm 1$, all the words in the word expansion of $e_i|\pi\rangle$ has the property that e_j can be moved the leftmost. (P4) follows from this observation.

Remark3: The cyclic operator σ acts on a state $|\pi\rangle$ as follows. Let us introduce the cyclic operator $\bar{\sigma}$ acting on an unrestricted path as $\bar{\sigma} : \pi \mapsto \pi' = \pi_2 \ldots \pi_N\pi_1$. The action of σ is given by

$$\sigma|\pi\rangle = C_{\bar{\sigma}\pi}|\bar{\sigma}\pi\rangle \tag{98}$$

with a certain constant $C_{\bar{\sigma}\pi}$. From $\bar{\sigma}^N = 1$, we normalize $\sigma^N = 1$, or equivalently, $\prod_{k=1}^{N} C_{\bar{\sigma}^k\pi} = 1$. To see why Eqn.(98) works, consider the action of σ on the state $|\pi^\Omega\rangle$. Note that $\sigma e_i|\pi^\Omega\rangle = e_{i-1}\sigma|\pi^\Omega\rangle$ and $e_{pk+q}|\pi^\Omega\rangle = \tau|\pi^\Omega\rangle$ holds for $0 \leq p \leq n-1$ and $1 \leq q \leq k-1$. The state $|\bar{\sigma}\pi^\Omega\rangle$ is the

only state that has the same convexity as $\sigma|\pi^{\Omega}\rangle$. Then we have $\sigma|\pi^{\Omega}\rangle \propto |\bar{\sigma}\pi^{\Omega}\rangle$. The action of σ on a state $|\pi\rangle = \prod_{i\in I}\check{L}_i(m_i)|\pi^{\Omega}\rangle$ is written as $\sigma|\pi\rangle = \prod_i \check{L}_{i-1}(m_i)\sigma|\pi^{\Omega}\rangle \propto |\bar{\sigma}\pi\rangle$. Eqn.(98) follows from these considerations.

Remark4: Although we are dealing with the affine Hecke algebra considered in Section 2, most of the above statements are also available to the Hecke algebra just by reducing the state space to only restricted paths. The cyclic operator is written in terms of the Hecke generators as $\sigma = t_{N-1}^{-1}\ldots t_1^{-1}$. In other words, the vector space spanned by states labelled by unrestricted paths is reducible in the sense of Definition 4.5 if we consider the Hecke algebra.

We have no relation like the cylindric relations for the case of the Hecke algebra and do not have the rhombus for $\check{L}_N$ since the algebra has no affine generator e_N. Afterall, the piling of rhombi stops when we have the path $1_n 2_n\cdots k_n$ where $l_n = \underbrace{l\cdots l}_{n}$. The case of restricted paths is considered in [23].

4.2.2 Relation to spin chain model

From the above construction of states of the A_k generalized model, we see a bridge between the A_k model and the spin chain model.

The spin representation of the generators of the affine Hecke algebra gives the R-matrix of the spin chain as in [16, 17, 26]. The transfer matrix of the $U_q(\mathfrak{gl}_k)$ spin chain is

$$T(u) = \mathrm{Tr}_0 R_{01}(u)R_{02}(u)\cdots R_{0L}(u), \tag{99}$$

where the trace is taken for the auxiliary quantum space indexed by 0. The Hamiltonian is given by [26]

$$\mathcal{H} = T^{-1}(u)\frac{dT(u)}{du}\Big|_{u=1} = \sum_{i=1}^{L-1}\mathcal{H}_{i,i+1} + \mathcal{H}_{L,1}, \tag{100}$$

where

$$\begin{aligned}
\mathcal{H}_{i,i+1} &= \frac{1}{q-q^{-1}}\mathcal{P}_{i,i+1}R'_{i,i+1} \quad 1\le i\le L-1, && (101)\\
\mathcal{H}_{L,1} &= \frac{1}{q-q^{-1}}\mathcal{P}_{L,1}\tilde{R}'_{L,1}, && (102)\\
R'_{ab} &= 2(\mathbb{I}\otimes\mathbb{I} - \mathcal{P}^q_{ab}) + (q+q^{-1})\mathcal{P}_{ab}, && (103)\\
\tilde{R}'_{ab} &= 2((\mathbb{I}\otimes\mathbb{I})_q - \mathcal{P}^q_{ab}) + (q+q^{-1})\mathcal{P}_{ab}. && (104)
\end{aligned}$$

where $\mathcal{P}, \mathcal{P}^q$ and $\tilde{\mathcal{P}}^q$ are permutations defined in Section 3. We focus on the eigenvector of the Hamiltonian $\mathcal{H}$.

Suppose that $\Psi(\mathbf{z})$ is the solution of the q-KZ equation (82) at $q = -\exp(\pi i/(k+1))$. Because of the commutation relation between the transfer matrix of the A_k generalized model and that of the spin chain, the solution Ψ in the homogeneous limit is also the eigenvector of the spin chain at

$q = -\exp(\pi i/(k+1))$. Spin chain models have nice properties at this special point. For instance, it is conjectured that the free-fermion part of the spectrum of the $SU_q(k+1)$ Perk-Schultz at $q = -\exp(\pi i/(k+1))$ is a consequence of nice properties of the inhomogeneous $SU_q(k)$ vertex model [28].

Let us discuss the relation between the states of the A_k generalized model and the vector space of the spin chain model. All the states of the A_k model are obtained by acting a sequence of e_i's on the product of the q-symmetrizer, $Y_q\text{-sym}$. From Proposition 3.2 in the spin representation, $|v_0\rangle$ (Eqn.30) is the only eigenvector of $Y_q\text{-sym}$ with non-zero eigenvalue. Together with Eqn.(40), the lowest state $|\pi^\Omega\rangle$ is identified with the vector $|v_0\rangle$ in the spin representation. All the other states are expressed in terms of vectors in the spin representation by multiplying $|v_0\rangle$ by a sequence of $\check{L}$-matrix in the spin representation. This is a natural generalization of Eqn.(89) and Eqn.(90). We are not able to write down the expression as simply as in the case of the XXZ spin chain. However, this allows us to write down the solution $\Psi(\mathbf{z})$ in terms of the spin representation.

The spin chain considered here is the model of spin-$(k-1)/2$. The total number of sites n_i with $S_z = (2i-k-1)/2$, $1 \le i \le k$ are conserved quantities. The state space of the A_k generalized model is the vector space of spin vectors with all $n_i = n$ in the spin representation. We can analyze the A_k generalized model in the spin representation by constructing words.

4.2.3 q-KZ equation and the sum rule

We solve the q-KZ equation following the method used in [18, 22]. The solution is supposed to be the one of the minimal degree. In particular, we consider the solution with q a root of unity, $q = -\exp(\pi i/(k+1))$ and $s = 1$.

q-KZ equation The q-KZ equation (82) at $(s, q) = (1, -\exp(\pi i/(k+1)))$ is rewritten as

$$t_i\Psi(\mathbf{z}) = (e_i - \tau)\Psi(\mathbf{z}), \qquad 1 \le i \le N \tag{105}$$

where $z_{N+1} = z_1$ and t_i acts on a polynomial $f(\mathbf{z}) := f(z_1, \dots, z_N)$ as

$$t_i f(\mathbf{z}) := \frac{qz_i - q^{-1}z_{i+1}}{z_{i+1} - z_i}(\tau_i - 1) f(\mathbf{z}) \tag{106}$$

and $\tau_i f(\dots, z_i, z_{i+1}, \dots) = f(\dots, z_{i+1}, z_i, \dots)$.

Consider the state which is not invariant under the action of e_i. The π-th element of (105) is $t_i\Psi_\pi(\mathbf{z}) = -\tau\Psi_\pi(\mathbf{z})$, or equivalently,

$$(qz_i - q^{-1}z_{i+1})\tau_i\Psi_\pi(\mathbf{z}) = (qz_{i+1} - q^{-1}z_i)\Psi_\pi(\mathbf{z}). \tag{107}$$

Since $\Psi_\pi(\mathbf{z})$ is supposed to be a polynomial, $\Psi_\pi(\mathbf{z})$ has a factor $(qz_i - q^{-1}z_{i+1})$.

Let $\tau_{i,i+l} := \tau_i \dots \tau_{i+l-2}\tau_{i+l-1}\tau_{i+l-2} \dots \tau_i$ be an exchange operator such that $\tau_{i,i+l} f(\dots, z_i, \dots, z_{i+l}, \dots) = f(\dots, z_{i+l}, \dots, z_i, \dots)$. Then, we have

$$\check{P}(z_i, \dots, z_{i+l})\Psi = \tau_{i,i+l}\Psi \tag{108}$$

where

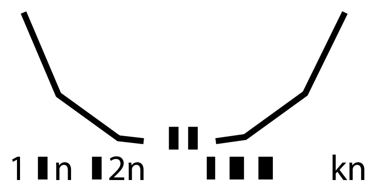
$$\begin{aligned}\check{P}(z_i,\dots,z_{i+l}) &= \check{R}_i(z_{i+l},z_{i+1})\dots\check{R}_{i+l-2}(z_{i+l},z_{i+l-1})\\ &\quad\cdot\check{R}_{i+l-1}(z_{i+l},z_i)\check{R}_{i+l+2}(z_{i+l-2},z_i)\dots\check{R}_i(z_{i+1},z_i). \end{aligned}\tag{109}$$

Consider a state π which is not invariant under the action of e_{i+j} for $0 \le j \le l-1$. Taking into account that $\check{R}_{i+l-1}(z_{i+l},z_i) \propto e_{i+l-1}$ if we set $z_{i+l} = q^{-2}z_i$, it is shown that

$$\tau_{i,i+l}\Psi_\pi(\mathbf{z})|_{z_{i+l}=q^{-2}z_i} = 0. \tag{110}$$

This means $\Psi_\pi(\mathbf{z})$ has a factor $(qz_i - q^{-1}z_{i+l})$. Therefore, in total $\Psi_\pi(\mathbf{z})$ has factors $\prod_{i\le m<n\le i+l}(qz_m - q^{-1}z_n)$.

Highest weight state The highest state π^0 of this model is given by the following path:



(111)

$\pi^0 = \{\pi_i|\pi_{pn+q} = k+1-p, 0 \le p \le k-1, 1 \le q \le n\}$. This highest weight is characterized by

$$e_{kn}\pi^0 = \tau\pi^0, \tag{112}$$

$$e_i\pi^0 \neq \pi^0, \quad \text{for } 1 \le i \le kn-1. \tag{113}$$

The number $\sharp\{i|e_i\pi^0 = \tau\pi^0\}$ is minimal. The highest state is invariant (up to a constant) only under the action of e_{kn}. The entry Ψ_{π^0} is written as

$$\Psi_{\pi^0} = \prod_{1\le i<j\le nk}(qz_i - q^{-1}z_j), \tag{114}$$

under the assumption of the minimal degree. The total degree of Ψ_{π^0} is $N(N-1)/2$ and the partial degree is $N-1$ for each z_i.

Recursive relation Let us fix an integer m and take a special parameterization of the form

$$z_{m+j} = q^{-2j}z, \qquad 0 \le j \le k-1. \tag{115}$$

This kind of specializations is called the wheel condition in the theory of symmetric polynomials [29]. The entry $\Psi_\pi(\mathbf{z})$ is non-vanishing only when π has the convex sequence $\pi_{m+j} = j+1$ for $0 \le j \le k-1$.

For an unrestricted path π of length $(n-1)k$, let $\varphi_{m,m+k-1}(\pi)$ be an embedded path of length nk where the convex sequence of length k is inserted between π_m and π_{m+1}. Then, we have the following recursion relation

$$\Psi_{\varphi_{m,m+k-1}(\pi)}(\mathbf{z})|_{(115)} = Cz^{\frac{1}{2}k(k-1)}\left(\prod_{\substack{1\le j\le N\\ j\neq m,\dots,m+k-1}}(qz_i - q^{-1}z)^k\right)\Psi_\pi(\mathbf{z}') \tag{116}$$

where $\mathbf{z}' = \mathbf{z}\backslash\{z_m, \ldots, z_{m+k-1}\}$ and C is some constant depending only q and N. To see this relation, suppose that $\Psi(\mathbf{z})$ is the minimal degree solution for N variables. The r.h.s of (116) satisfies the q-KZ equation for $N-k$ variables. This assures that $\Psi(\mathbf{z}')$ is also the solution of the q-KZ equation. Note that the total degree and partial degrees with respect to all z_i are consistent.

Razumov-Stroganov point and the sum rule We define the simultaneous eigen covector v satisfying

$$
\begin{aligned}
ve_i &= \tau v, && (117)\\
v\sigma &= v, && (118)
\end{aligned}
$$

or we may write as $v\check{R}_{ii+1} = v$ for all i. The existence of v requires that q be a root of unity. Together with the vanishing condition of the q-symmetrizer (2), we should have $U_k(\tau) = 0$, *i.e.*, we should take the Razumov-Stroganov (RS) point $q = -\exp\left(\frac{i\pi}{k+1}\right)$. In the below, q is taken as this RS point.

The sum rule is the formula for the weighted sum, $W(\mathbf{z}) = v\cdot\Psi = \sum_\pi v_\pi \Psi_\pi(\mathbf{z})$. One can show that $W(\mathbf{z})$ is a homogeneous and symmetric polynomial with respect to all the variables z_i, which is led from the fact that the polynomial $\Psi_{\pi^0}(\mathbf{z})$ is homogeneous and the actions of t_i preserve this property. Since $\tau_i W(\mathbf{z}) = v\cdot\tau_i\Psi = v\cdot\check{R}_i\Psi = W(\mathbf{z})$, $W(\mathbf{z})$ is a symmetric polynomial. From the recursive relation (116), we have the recursive relation

$$
W(\mathbf{z})|_{(115)} = Cz^{\frac{1}{2}k(k-1)}\left(\prod_{\substack{1\le j\le N\\ j\neq m,\ldots,m+k-1}} (qz_i - q^{-1}z)^k\right) W(\mathbf{z}'). \tag{119}
$$

The total degree and partial degree of $W(\mathbf{z})$ are $\binom{N}{2}$ and $N-1$ respectively. From the above observation and Proposition 5.2, we can show that the sum $W(\mathbf{z})$ is written in terms of Schur functions $s_\lambda(\mathbf{z})$ as

$$
W(\mathbf{z}) = (\text{const.}) \prod_{l=0}^{k-1} s_{Y^n_{k,l}}(z_1, \ldots, z_{nk}) \tag{120}
$$

with an appropriate overall normalization. Here, the Young diagrams are $Y^n_{k,l} := \delta(n^l, n-1^{k-l})$ with

$$
\delta(n^l, n-1^{k-l}) = (\underbrace{n,\cdots,n}_{l}, \underbrace{n-1,\cdots,n-1}_{k}, \underbrace{n-2,\cdots,n-2}_{k}, \cdots, \underbrace{1,\cdots,1}_{k}). \tag{121}
$$

Remark: When $k=2$, we have

$$
W(z_1,\cdots,z_{2n}) = (\text{const.}) s_{Y^n_{2,0}}(z_1,\cdots,z_{2n}) s_{Y^n_{2,1}}(z_1,\cdots,z_{2n}). \tag{122}
$$

We reproduce the sum for the $O(1)$ loop model on a cylinder [22, 14].

4.3 Relation to special solutions of the q-KZ equation

In this subsection, we show that the eigenvector of the transfer matrix of the A_k generalized model at the Razumov-Stroganov point is viewed as the special solution of the q-KZ equation at $(q, s) = (-\exp(i\pi/(k+1)), 1)$ in [15].

In Section 4 of [15], special solutions of the q-KZ equation were constructed from the non-symmetric Macdonald polynomial [30] through the action of the affine Hecke algebra. Consider the q-KZ equation on the spin representation instead of on the space of paths. Let l and r be positive integers such that $1 \le l \le \min\{N-1, k\}$, $r \ge 2$ and $l+1$ and $r-1$ are coprime. We take the specialization

$$q^{2(l+1)} s^{-(r-1)} = 1. \tag{123}$$

An element $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$ is said to be admissible if λ satisfies

$$\lambda_i^+ - \lambda_{i+l}^+ \le r-1 \qquad \text{for any } 1 \le i \le n-l, \tag{124}$$

$$\lambda_i^+ - \lambda_{i+l}^+ = r-1 \qquad \text{only if } w_\lambda^+(i) < w_\lambda^+(i+k). \tag{125}$$

Here, λ^+ is the unique dominant in $\mathfrak{S}_N\lambda$, *i.e.* $\lambda_1^+ \ge \lambda_2^+ \ge \dots \ge \lambda_N^+$, and w_λ^+ is the shortest element in $\mathfrak{S}_N$ such that $w_\lambda^+\lambda^+ = \lambda$. Let $\mu \in \mathbb{Z}^N$ be an element constructed from an dominant element $a \in \mathbb{Z}^l$ (see Section 4.3 in [15]). Then, the solution of the q-KZ equation of level $\frac{l+1}{r-1} - N$ is created from the non-symmetric Macdonald polynomial E_μ with the specialization (123) and an admissible μ (lemma 4.5 and Theorem 4.6 in [15]).

We want to find the solution of the q-KZ equation on the spin basis, which is dual to the eigenvector of the A_k generalized model. The eigenvector of the transfer matrix of the A_k generalized model at the Razumov-Stroganov point is characterized by the highest weight state (114). A monomial $\prod_{j=1}^N z_j^{N-j}$ is the dominant one in the expansion of the r.h.s. of Eqn.(114). Once the higest weight state is fixed, the eigenvector of the A_k generalized model is uniquely determined.

We restrict ourselves to the special solution of the q-KZ equation with the level $1 - k + \frac{1}{k}$ on the spin basis at the specializations

$$(l, r) = (k, k+1), \tag{126}$$

$$a = (nk-1, \dots, k(n-1)) \in \mathbb{Z}^k, \tag{127}$$

$$(q, s) = (-\exp(i\pi/(k+1)), 1). \tag{128}$$

Note that the dominant monomial of the solution is $\prod_{j=1}^N z_j^{N-j}$.

From the construction, a state of the A_k generalized model is written as a linear combination of the spin basis (see Section 4.2.1 and 4.2.2). Therefore, a suitable linear comibination of the above solution of the q-KZ equation on the spin basis gives the eigenvector of the transfer matrix of the A_k generalized model at the RS point. The eigenvector of the A_k generalized model and the special solution of the q-KZ equation share the same dominant monomial. Together with the uniqueness of the eigenvector of the A_k generalized model, the eigenvector of the A_k generalized model at the Razumov-Stroganov point coincides with the special solution of the q-KZ equation characterized by Eqns.(126)-(128). When $k = 2$, we can see the solution with $(l, r) = (2, 3)$ as in [14].

5 Recursive Relation for Schur Functions

Let us denote Young diagrams by $Y^n_{k,l} = \delta(n^l, n-1^{k-l})$ and $Y^1_{k,l} = \delta(1^l, 0^{k-l})$ where

$$\delta(n^l, n-1^{k-l}) = (\underbrace{n, \cdots, n}_{l}, \underbrace{n-1, \cdots, n-1}_{k}, \underbrace{n-2, \cdots, n-2}_{k}, \cdots, \underbrace{1, \cdots, 1}_{k}). \tag{129}$$

Proposition 5.1. *When $q = -\exp(\frac{i\pi}{k+1})$, the principal specialization of $s_{Y^1_{k,l}}$ is given by*

$$s_{Y^1_{k,l}}(1, q^2, \cdots, q^{2(k-1)}) = (-1)^l q^{-2l}. \tag{130}$$

Proposition 5.2. *Set $q = -\exp(\frac{i\pi}{k+1})$. The following recursion relation for the Schur function $s_{Y^n_{k,l}}$ holds:*

$$s_{Y^n_{k,l}}(z_1, \cdots, z_{nk})\Big|_{wheel} \;=\; (-)^l q^{-2l} z^l \prod_{i=1}^{(n-1)k} (z_i - q^{2k} z) s_{Y^{n-1}_{k,l}}(z_1, \cdots, z_{k(n-1)}) \tag{131}$$

where the wheel condition (the specialization of variables in Eqn.(131), see also [29]) is such that $z_{j_m} = q^{2(j_m-1)}z$ for $1 \le m \le k$ and $j_m < j_{m+1}$.

Further, if we write as $S^n(\mathbf{z}) := \prod_{l=0}^{k-1} s_{Y^n_{k,l}}$, we have the following recursive relation from the proposition 5.2,

$$S^n(\mathbf{z})|_{z_{k(n-1)+j}=q^{2(j-1)}z} = (-)^{k(k+1)/2} q^2 z^{k(k-1)/2} \prod_{i=1}^{(n-1)k} (z_i - q^{2k} z)^k S^{n-1}(\mathbf{z}') \tag{132}$$

where $\mathbf{z}' = \mathbf{z} \backslash \{z_{k(n-1)+1}, \cdots, z_{nk}\}$.

Proof. Since a Schur function $s_\lambda(x)$ is a symmetric function with respect to all the variables x_i, we consider only the following specialization of variables without loss of generality:

$$z_{k(n-1)+j} = q^{2(j-1)} z, \qquad 1 \le j \le k. \tag{133}$$

Since the number of boxes in the first column of the Young diagram $Y^n_{k,l}$ is $k(n-1)+l$, we have at least a factor z^l when we take the specialization (133). Around $z = 0$, the l.h.s of (131) is approximated as

$$s_{Y^n_{k,l}}(z_1, \cdots, z_{nk})\Big|_{z_{k(n-1)+j}=q^{2(j-1)}z} \propto z^l \prod_{i=1}^{(n-1)k} z_i \cdot s_{Y^{n-1}_{k,l}}(z_1, \cdots, z_{k(n-1)}) + O(z^{l+1}). \tag{134}$$

The maximal and minimal degrees with respect to z of the first term of the r.h.s. of Eqn.(134) is $k(n-1)+l$ and l respectively.

A Schur function of m-variables has an expression in terms of determinants as

$$s_\lambda(z_1, \cdots, z_m) = \frac{\det z_i^{\lambda_j+m-j}}{\det z_i^{m-j}} = \frac{\det z_i^{\lambda_j+m-j+l+1}}{\prod z_i^{l+1} \det z_i^{m-j}}. \tag{135}$$

For $\lambda = Y^n_{k,l}$ and $m = nk$, the sequence $\lambda'_j = \lambda_j + m - j + l + 1$ has the form

$$\lambda' = (l, \cdots, 1, \underbrace{k, k-1, \cdots, 1}\cdots, \underbrace{k, \cdots, 1}) \mod k+1. \tag{136}$$

Note that there is all λ'_j is not a multiple of $(k+1)$.

Specialize k variables as $z_{k(n-1)+j} = q^{2(j-1)}z$ $(1 \le j \le k)$ and set $z_i = q^{2k}z$ for $i \neq k(n-1)+j, 1 \le \forall j \le k$ in the determinant expression (135). We find that $k+1$ row-vectors in the det of the numerator of (135), say v_r, $0 \le r \le k$, are such that $(v_r)_i = q^{2r\lambda'_i}z^{\lambda'_i}$. These $k+1$ row-vectors are not linearly independent, since we have

$$\sum_{r=0}^{k}(v_r)_i = \left(\sum_{r=0}^{k} q^{2r}\right) q^{\lambda'_i} z^{\lambda'_i} = 0 \tag{137}$$

where we have used the relation $\sum_{r=0}^{k} q^{2r} = 0$ for $q = -\exp(\pi i/(k+1))$. The determinant turns to be zero under the above specialization of $k+1$ variables. By the symmetry of the variables in the Schur function, we find that the l.h.s of (131) has a factor $\prod_{1\le i\le n(k-1)}(z_i - q^{2k}z)$. Together with (134), we may write as

$$s_{Y^n_{k,l}}(z_1, \cdots, z_{nk})\Big|_{z_{k(n-1)+j}=q^{2(j-1)}z} \propto z^l \prod_{i=1}^{(n-1)k}(z_i - q^{2k}z) \cdot s_{Y^{n-1}_{k,l}}(z_1, \cdots, z_{k(n-1)}). \tag{138}$$

Actually, the total degree and partial degree of z_i in the both sides of (138) coincide. The prefactor $(-)^l q^{-2l} = s_{Y^1_{k,l}}(1, q^2, \cdots, q^{2(k-1)})$ is checked by collecting the terms with the lowest degree in z (see Proposition 5.1). □

6 Conclusion

In this paper we have defined and studied the A_k generalized model of the $O(1)$ loop model on a cylinder by using the representation of the affine Hecke algebra. The affine Hecke algebra is characterized by extra novel vanishing conditions, the cylindric relations. Two representations of the algebra have been given; the first one is based on the spin representation, and the other is based on states of the A_k generalized model. These two representation are connected by the word representation of states of the A_k generalized model. We have established an explicit way of constructing states of the A_k generalized model by the use of the rhombus tiling. We have shown that the Yang-Baxter equation and q-symmetrizers are depicted as hexagons and polygons, respectively. The meaning of the cylindric relations is clearly seen in the graphical depiction of the A_k generalized model. The cylindric relations for the affine Temperley-Lieb algebra implies that a loop surrounding the cylinder returns a weight τ. For the A_k generalized model, a "band" consisting of rhombi surrounding the cylinder gives a certain weight in terms of the second kind of Chebyshev polynomials.

We have considered the eingenvector of the transfer matrix of the A_k generalized model at the Razumov-Stroganov point, $q = -\exp(\pi i/(k+1))$. It has been found that this eigenvector

coincides with the special solution of the q-KZ equation of level $1+\frac{1}{k}-k$ at $q=-\exp(\pi i/(k+1))$ and $s=1$. We have examined the sum rule for the A_k generalized model on a cylinder and shown the formula is written in terms of the product of k Schur functions. The obtained sum rule includes the sum rule for the $O(1)$ loop model on a cylinder when $k=2$.

There are still some open problems. It was shown that the sum rules for the $O(1)$ loop models with various boundary conditions are related to exactly solvable models with symmetries, alternating sign matrices with certain symmetries, or total numbers of the plane partitions with symmetries. We expect that the sum rule for the A_k generalized model on a cylinder may also relate to those objects. In the case of $k=2$, the total number of half-turn symmetric alternating sign matrices appeared in this context [22]. The method used in this paper is applicable to the Hecke algebras of other types. We hope to come back to these issues in the future.

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Appendix A

A.1

In this appendix, we will show that a class of $C_{i,\pi,\pi'}$ (see (P1-4) in Section 4) is equal to 1. Let us recall that the coefficient $C_{i,\pi,\pi'}$ may be non-zero when $\pi_i \leq \pi_{i+1}$.

We introduce a sequence of $\check{L}$-matrices as

$$\mathcal{L}_{i+1,i+l}(m)=\check{L}_{i+1}(m)\check{L}_{i+2}(m+1)\cdots\check{L}_{i+l}(m+l-1), \tag{139}$$

and $\mathcal{L}_{i,j}=1$ if $i>j$. Let $\mathcal{B}$ be a word representation corresponding a state $|\pi\rangle$ with $\pi_{i+l+1}<\pi_i<\pi_{i+1}<\ldots<\pi_{i+l}$ for $l\geq 1$. We consider a word of the form, $\mathcal{B}^{'}:=\mathcal{L}_{i+1,i+l}(m^{'})\mathcal{B}$.

Proposition A.1. *The action of $\check{L}_i(m)$ on $\mathcal{B}^{'}$ is given by*

$$\check{L}_i(m)\mathcal{B}^{'}=\mathcal{L}_{i,i+l}(m)\mathcal{B}+\mathcal{L}_{i+2,i+l}(m)\mathcal{B}. \tag{140}$$

Proof. We use the method of induction. we assume

$$\check{L}_{i+l'-1}(m)\mathcal{L}_{i+l',i+l}(m)\mathcal{B}=\mathcal{L}_{i+l'-1,i+l}(m)\mathcal{B}+\mathcal{L}_{i+l'+1,i+l}(m)\mathcal{B} \tag{141}$$

for $1\leq l^{'}\leq l-1$. From the above assumption, we have

$$\begin{aligned}\check{L}_i(m)\mathcal{B}^{'} &= \check{L}_i(m)\mathcal{L}_{i+1,i+l-1}(m)\check{L}_{i+l}(m+l-1)\mathcal{B}\\ &= (\mathcal{L}_{i,i+l-1}(m)+\mathcal{L}_{i+2,i+l-1}(m)\check{L}_{i+l}(m+l-1)\mathcal{B}\\ &= \mathcal{L}_{i,i+l}(m)\mathcal{B}+\mathcal{L}_{i+2,i+l}(m)\mathcal{B}\\ &\quad +(\Delta_{m+l-1}\mathcal{L}_{i,i+l-1}(m)-\Delta_{m+l-2}\mathcal{L}_{i+2,i+l-1}(m))\mathcal{B}\end{aligned} \tag{142}$$

where we used the word $\check{L}_{i+l}(m+l-1)\mathcal{B}$ satisfies $\pi_{i+l} < \pi_1 < \ldots < \pi_{i+l-1}$ and $\Delta_k = \mu_k - \mu_{k-1}$. By using $e_j\mathcal{B} = \tau\mathcal{B}$ for $i \le j \le i+l-1$, the third term in (142) is

$$\begin{aligned} \text{the 3-rd term} &= \prod_{r=1}^{l-2} \frac{1}{\mu_{m+r-1}} \left(\frac{\Delta_{m+l-1}}{\mu_{m+l-2}\mu_{m+l-1}} - \Delta_{m+l-2} \right) \mathcal{B} \\ &= 0 \end{aligned} \tag{143}$$

where we used the relation $\frac{1}{\mu_k\mu_{k-1}}\Delta_k = \Delta_{k-1}$. We finally obtain Eqn. (140) holds true by induction. □

Set $m = 1$ in Eqn. (140). Together with the construction of states considered in Section 4.2, we have the following corollary:

Corollary A.2. *Suppose that a state $|\pi^0\rangle$ is equivalent to $\mathcal{B}'$ in the word representation and $\pi^0_{i+1} \le \pi^0_{i-1}$. We have*

$$e_i|\pi^0\rangle = |\pi^1\rangle + |\pi^2\rangle \tag{144}$$

where

$$\pi^1 = \begin{cases} \pi^1_i = \pi^0_{i+1}, \pi^1_{i+1} = \pi^0_i, \\ \pi^1_j = \pi^0_j \quad \text{for} \quad j \ne i, i+1, \end{cases} \qquad \pi^2 = \begin{cases} \pi^2_{i+1} = \pi^0_{i+2}, \pi^2_{i+2} = \pi^0_{i+1}, \\ \pi^2_j = \pi^0_j \quad \text{for} \quad j \ne i+1, i+2. \end{cases} \tag{145}$$

A.2

A.2.1

We have six states in the case of $k = N = 3$. The word representation of states is listed as

word	e_3Y_2	$Z_{2,3}Y_2$	$Z_{13}Y_2$	$e_1Z_{2,3}Y_2$	$e_2Z_{1,3}Y_2$	Y_2
path	321	312	231	132	213	123

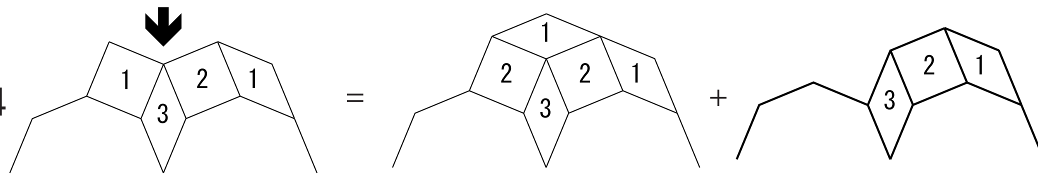
where $Z_{i,j} = e_ie_j - 1$ and $Y_2 = e_1e_2e_1 - e_1$. We obtain the representation of the generators:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \tau^2-1 & 0 & \tau & 0 & 1 & 0 \\ 0 & 1 & 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \tau^2-1 & \tau \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \tau^2-1 & \tau & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \tau & 0 \\ 1 & 0 & 0 & \tau^2-1 & 0 & \tau \end{pmatrix},$$

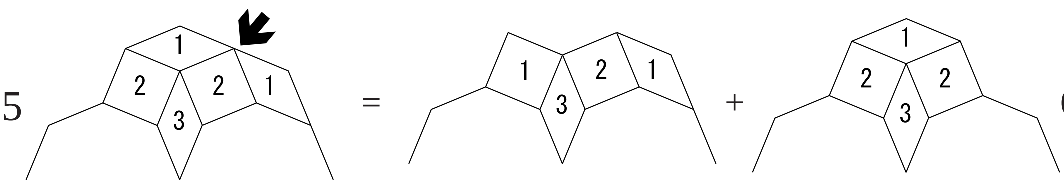
$$e_3 = \begin{pmatrix} \tau & 0 & 0 & 0 & 0 & 1 \\ 0 & \tau & 0 & 1 & \tau^2-1 & 0 \\ 0 & 0 & \tau & \tau^2-1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A.2.2

Two examples how the generator of the affine Hecke algebra acts on a state. We consider the case where $k = 4$ and $n = 2$.

e_4 

(146)

e_5 

(147)

The bold arrows indicate where the generators act. The state of the l.h.s. of Eqn. (146) is an example of states which do not satisfy the zero-sum rule on the top vertices of Y_q-sym. We see that the properties (P2) and (P4) are satisfied.

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